BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

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F. N. COLE E. W. BROWN VIRGIL SNYDER
ALEXANDER ZIWET D. E. SMITH
J. W. YOUNG W. B. FORD

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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

NOTE ON FERMAT'S NUMBERS.

BY DR. J. C. MOREHEAD AND MR. A. E. WESTERN.

(Read before the Chicago Section of the American Mathematical Society, April 9, 1909.)

In June, 1658, Fermat wrote to Sir Kenelm Digby a letter,* in which, after referring to certain theorems proved by him, which he might propose to Viscount Brouncker and John Wallis, in order to give them something to do, he said that, instead of these theorems, he would submit to them, as problems, some theorems which he admitted he could not prove, though he was convinced of their truth. The first of these problems was to prove that $2^{2^n}+1$ is a prime, and he gave as examples the numbers corresponding to $n=1, 2, 3, 4$, which in fact are primes. Fermat challenged his English friends to furnish a proof of this proposition, which was certainly very beautiful, and which he believed was true. He added that perhaps the proof would give the key to penetrate all the mystery of prime numbers.†

As is well known, the theorem is untrue for many values of n.‡ In 1905 Dr. Morehead read before the AMERICAN MATHEMATICAL SOCIETY a "Note on Fermat's numbers," ‡ which stated the result of a calculation proving that $F_7=2^{128}+1$ is composite. Mr. Western had independently performed the same calculation, and communicated the result almost simul-

* Pierre de Fermat, Œuvres, Paris, 1896, vol. 2, p. 405 (in Latin); vol. 3, p. 315 (French translation).

† That Fermat attached great importance to this theorem is further evidenced by the fact that he referred to it in six other letters and papers written between 1640 and 1659. Cf. Œuvres, vol. 1, p. 131; vol. 2, pp. 206, 207, 212, 309, 434.

‡ W. W. R. Ball, Mathematical Recreations and Problems, 2d ed., 1892, p. 26. *Proc. Lond. Math. Soc.*, Series 2, vol. 1 (1904), p. 175. BULLETIN, vol. 11, p. 543, and vol. 12, p. 449.

taneously to the London Mathematical Society;* and it was found that the two calculations were in exact agreement. There is therefore no doubt that F_7 is composite, but the actual factors are unknown. The authors have since carried out a similar calculation for F_8, each doing a half of the whole work. As this is probably by far the largest calculation in connection with the theory of numbers which has been yet performed, some particulars of the methods employed may be of interest.

The test employed by the authors in the case both of F_7 and F_8 depends on the following theorem:

If $a^x \equiv 1 \pmod{p}$ *is true when* $x = p-1$, *but is not true when* x *is any factor of* $p-1$, *then* p *is prime.*

This theorem appears to have been first given by Lucas in 1876, and is called by him the reciprocal of Fermat's theorem.† The proof is very simple; if p is composite, then $\phi(p) < p-1$; and if the assumption of the theorem is satisfied, then $a^x \equiv 1 \pmod{p}$ for both $x = \phi(p)$ and $x = p-1$, and therefore also for $x = \delta$, the greatest common divisor of $\phi(p)$ and $p-1$, which contradicts the assumption. Therefore, under the conditions of the theorem, p cannot be composite.

Applying this to the case of Fermat's numbers, and taking $a = 3$, it is clear that if F_n is prime,

$$3^{\frac{1}{2}(F_n-1)} \equiv -1 \pmod{F_n},$$

for $F_n \equiv -1 \pmod 3$, and so 3 is a quadratic non-residue of F_n. And since $\frac{1}{2}(F_n-1)$ is a power of 2, no value of x less than $\frac{1}{2}(F_n-1)$ will make

$$3^x \equiv -1 \pmod{F_n}.$$

Therefore by Lucas's theorem, if

$$3^{\frac{1}{2}(F_n-1)} \equiv -1 \pmod{F_n},$$

F_n is prime, and if not, F_n is composite.

The index of 3 in this congruence may be reduced; for

$$-1 \equiv 2^{2^n} \pmod{F_n},$$

and we therefore obtain, by successive extractions of the square root,

$$3^{2^{(2^n-n-1)}} \equiv 2^x \pmod{F_n},$$

* *Proc. Lond. Math. Soc.*, ser. 2, vol. 3, p. xxi.

† Lucas, Théorie des Nombres; Paris, 1891, p. 441.

where x is odd. Now

$$2 \equiv 2^{2^{n-1}}(2^{2^{n-1}} - 1)^2 \pmod{F_n},$$

and so

$$(A) \qquad 3^{z(2^n-n-2)} \equiv \pm 2^y(2^{2^n-1} \pm 1) \pmod{F_n},$$

where y may be taken to be between 0 and 2^{n-1}.

Applying this criterion to F_8, it is necessary to calculate the residue of the 2^{246}th power of 3 (mod F_8).

Mr. Western's method was as follows. Having obtained

$$3^{2^s} \equiv \Sigma a_n x^n \pmod{F_8} \quad (n = 0, 1, \cdots, 7),$$

where $x = 2^{32}$, and $a_0, a_1, \cdots, a_7$ are positive or negative numbers less than $\frac{1}{2}x$ (*i. e.* containing at most 10 digits), the residue on the right is squared; since $x^8 \equiv -1$, we obtain

$$3^{2^{s+1}} \equiv \Sigma A_n x^n \pmod{F_8},$$

where

$$A_0 = a_0^2 - 2a_1a_7 - 2a_2a_6 - 2a_3a_5 - a_4^2,$$

$$A_1 = 2(a_0a_1 - a_2a_7 - a_3a_6 - a_4a_5), \quad \text{etc.};$$

thus division by F_8 is performed simultaneously with the operation of squaring. The numbers $A_0, A_1, \cdots$ are each less than $2x^2$ (and generally less than x^2); each A is then divided by x, giving

$$A_r = b_r + xc_r,$$

where

$$|b_r| < \tfrac{1}{2}x, \quad |c_r| < 2x,$$

and then

$$3^{2^{s+1}} \equiv (b_0 - c_7) + (b_1 + c_0)x + \cdots + (b_7 + c_6)x^7.$$

The coefficients in this are then adjusted, so as to make each of them numerically less than $\frac{1}{2}x$, and we finally get

$$3^{2^{s+1}} \equiv \Sigma a'_n x^n \pmod{F_8}.$$

The whole process is then repeated. The calculations of A_r, b_r, c_r, were wholly performed on a 10-figure arithmometer, and the b_r and c_r alone were written down.

In such a calculation as this, in which a single error would vitiate all the subsequent work, care must be taken to detect any mistakes in each stage, before proceeding to the next stage. The test employed by Mr. Western was

$$\sum a' \equiv (a_0 + a_1 + a_2 + a_3)^2 - (a_4 + a_5 + a_6 + a_7)^2$$
$$+ 2(a_0 + a_1 + a_2 + a_3)(a_4 + a_5 + a_6 + a_7) - 4a_3(a_5 + a_7)$$
$$- 4a_6(a_2 + a_3) - 4a_7(a_1 + a_2) - 2(c_7 + d) \pmod{k},$$

where d is the multiplier of x^8 carried forward from the last term $(b_7 + c_6)x^7$ to the first term, and k is any factor of $x - 1$. The values of k used were 3, 5, and 17.

Dr. Morehead expressed the residue of $3^{2^{126}}$, calculated by Mr. Western, in the form $ax + b$, where $x = 2^{128}$ and $|a|$, $|b| < \frac{1}{2}x$; then squaring and replacing x^2 by -1, he obtained

$$3^{2^{127}} \equiv 2abx + (b - a)(b + a) \pmod{F_8}.$$

The products $2ab$, $(b - a)(b + a)$ were then calculated and expressed, by division by x, in the forms $\alpha x + \alpha'$, $\beta x + \beta'$ respectively, so that

$$2abx + (b - a)(b + a) = (\alpha x + \alpha')x + \beta x + \beta'$$
$$\equiv (\alpha' + \beta)x + \beta' - \alpha \pmod{F_8},$$

$\alpha, \alpha', \beta, \beta'$ having been so adjusted that $|\alpha' + \beta|$, $|\beta' - \alpha| < \frac{1}{2}x$. Thus was obtained

$$3^{2^{127}} \equiv a'x + b' \pmod{F_8},$$

and, by repetition of this process, the residues of

$$3^{2^{128}}, 3^{2^{129}}, \cdots, 3^{2^{246}}.*$$

The test formula (A) shows that if F_8 were prime we should have, in the residue of $3^{2^{246}}$, $|a| = |b|$. This residue was found to be

$$(107\ 2093\ 3158\ 0550\ 8180\ 4331\ 6350\ 5866\ 1999\ 4098)x$$
$$+ (\ 34\ 1778\ 3881\ 0697\ 6545\ 8021\ 2127\ 5588\ 1034\ 4254),$$

and therefore F_8 is composite.

This result is especially interesting as completing a chain of

* The part of the calculation carried out by Dr. Morehead was checked at each stage by applying the test

$$-ab + b^2 - a^2 \equiv 4a' + b' - \alpha \pmod{9},$$

and similar tests for the moduli 11, 19, 41, 101. Two 8-figure calculating machines were used together for the greater part of the calculation and checking at each stage.

five composite Fermat numbers, F_5, F_6, F_7, F_8, F_9, following the first five (and only known) Fermat primes, 3, 5, 17, 257, 65537, and as leaving F_{10}(a 309-place number) the smallest Fermat number whose status is unknown. All the Fermat numbers $F_5, \cdots, F_{12}$, except F_{10}, are now known to be composite.

If ζ is a primitive 2^{n+1}th root of 1, so that $\zeta^{2^n} = -1$, F_n is the norm of $2 - \zeta$ in the field of ζ. Accordingly the problem of factoring F_n is the same as that of factoring $2 - \zeta$. It is possible that Fermat may have observed this, and may have assumed that it was improbable that $2 - \zeta$ could have complex factors of the form

$$a_0 + a_1\zeta + \cdots + a_{2^n-1}\zeta^{2^n-1}.$$

In fact, when $n \geqq 2$, the field contains an infinite number of units, and so $2 - \zeta$ can be expressed in an infinite number of ways as the product of two complex numbers of the field, one of these being a unit, and the other having F_n as its norm. But whether, for any value of n, $2 - \zeta$ can be expressed as the product of two actual numbers of the field, neither being a unit, is not known. It seems probable that the prime factors of $2 - \zeta$, if any, are always ideals.

Actual complex numbers containing the complex factors of known factors of Fermat's numbers may be easily found; for instance, ζ being a primitive 64th root of 1, and p being 641, the smaller factor of F_5,

$$p = 1 + 2^7 + 2^9.$$

But $2 \equiv \zeta \pmod{\pi}$, π being one of the prime factors of p in the field of ζ, so

$$1 + \zeta^7 + \zeta^9 \equiv 0 \pmod{\pi}.$$

The norm of $1 + \zeta^7 + \zeta^9$ is 193.641, as shown by Reuschle,* who states that the prime factors of 641 are ideal.

Again the smaller factor of F_6 is

$$p = 274{,}177 = 1 - 2^8 - 2^{12} + 2^{14} + 2^{18},$$

so ζ being as before, and π being a prime factor of p in the field of ζ, we have $4 \equiv \zeta \pmod{\pi}$, and so,

$$1 - \zeta^4 - \zeta^6 + \zeta^7 + \zeta^9 \equiv 0 \pmod{\pi}.$$

* Tafeln complexer Primzahlen, p. 455.

The norm of the number on the left is found to be p. It seems impracticable to determine whether or not p has actual prime factors in the field of 128th roots of 1, but this is very improbable, as the class number in that field is a multiple of 21,121.*

The use of complex numbers appears to be of no assistance in the problem of determining whether F_n is prime or composite.

AN EXTENSION OF CERTAIN INTEGRABILITY CONDITIONS.

BY PROFESSOR J. EDMUND WRIGHT.

SUPPOSE there are n functions $a_1, a_2, \cdots, a_n$ of n independent variables $x_1, x_2, \cdots, x_n$, satisfying the conditions

$$\frac{\partial a_p}{\partial x_q} - \frac{\partial a_q}{\partial x_p} = 0$$

for all values of p and q. It is well known that the functions a must all be first derivatives of a single function V. Similarly, if there are $\frac{1}{2}n(n+1)$ functions a_{pr} such that $a_{pr} = a_{qp}$, satisfying the relations

$$\frac{\partial a_{pq}}{\partial x_r} = \frac{\partial a_{pr}}{\partial x_q}$$

for all values of p, q, r, then the a's must be second derivatives of a single function.

The following question arises in connection with an application of the theory of invariants of quadratic differential forms:

Suppose there are $n(n+1)$ functions H_{pq}, K_{pq} such that $H_{pq} = H_{qp}$, $K_{pq} = K_{qp}$, satisfying the conditions

$$\frac{\partial}{\partial x_r}(H_{pq}) + K_{pq}\frac{\partial Y}{\partial x_r} = \frac{\partial}{\partial x_p}(H_{qr}) + K_{qr}\frac{\partial Y}{\partial x_p},$$

for all values of p, q, r; Y being a given function of the variables; what are the conditions on the functions H, K?

We first consider the case of $2n$ functions $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n$, satisfying the conditions

* Reuschle, Tafeln, p. 461.

$$(1) \qquad \frac{\partial a_p}{\partial x_q} - \frac{\partial a_q}{\partial x_p} = b_q \frac{\partial Y}{\partial x_p} - b_p \frac{\partial Y}{\partial x_q}.$$

Take three equations of the type (1), those for (p, q), (q, r), (r, p), differentiate the first with respect to r, the second with respect to p, the third with respect to q, and add. The quantities a are eliminated, and we have the result

$$(2) \qquad (b_{qr} - b_{rq})Y_p + (b_{rp} - b_{pr})Y_q + (b_{pq} - b_{qp})Y_r = 0,$$

where additional suffixes denote differentiation.

Now the equations (1) are unaltered if we replace b_p by $b'_p + \lambda Y$, where λ is an arbitrary function of the variables, and functions b and λ can be determined to satisfy the two equations

$$b_1 = \lambda Y_1 + \frac{\partial b}{\partial x_1}, \qquad b_2 = \lambda Y_2 + \frac{\partial b}{\partial x_2},$$

for elimination of λ gives a single equation for b, and any solution of this, combined with one of the above equations serves to determine λ.

We may thus in equations (1), (2), assume b_p replaced by b'_p, where b'_1 and b'_2 are first derivatives of a function b. Also we write

$$b'_p - \frac{\partial b}{\partial x_p} = b''_p.$$

In equation (2) give p, q, r, the values 1, 2, 3. It becomes precisely

$$J\left(\begin{matrix}\lambda, & b''_3 \\ x_1, & x_2\end{matrix}\right) = 0,$$

and therefore b''_3 is a function of $Y, x_3, x_4, \cdots, x_n$ only.

We can therefore find a function $F(Y, x_3, x_4, \cdots, x_n)$ such that $b''_3 = (\partial F/\partial x_3)_0$ where the suffix indicates that Y is kept constant. Hence

$$b''_3 = \frac{\partial F}{\partial x_3} - \frac{\partial F}{\partial Y} Y_3;$$

also

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial Y} Y_1, \qquad \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial Y} Y_2,$$

and therefore

$$b_1 = \left(\lambda - \frac{\partial F}{\partial Y}\right) Y_1 + \frac{\partial}{\partial x_1}(b + F),$$

$$b_2 = \left(\lambda - \frac{\partial F}{\partial Y}\right) Y_2 + \frac{\partial}{\partial x_2}(b + F),$$

$$b_3 = \left(\lambda - \frac{\partial F}{\partial Y}\right) Y_3 + \frac{\partial}{\partial x_3}(b + F),$$

or, changing the notation, we have found functions b and λ such that $b_p = \lambda Y_p + b'_p$, and $b'_p = \partial b / \partial x_p$ for $p = 1, 2, 3$.

If we now apply (2) for the three sets of values (1, 2, 4), (2, 3, 4), (3, 1, 4), we get

$$J\begin{pmatrix} Y, b''_4 \\ x_1, x_2 \end{pmatrix} = 0, \quad J\begin{pmatrix} Y, b''_4 \\ x_2, x_3 \end{pmatrix} = 0, \quad J\begin{pmatrix} Y, b''_4 \\ x_3, x_1 \end{pmatrix} = 0,$$

and hence b''_4 is a function of $Y, x_4, x_5, \cdots, x_n$ only. As before we may modify λ and b, so as to make $b'_p = \partial b / \partial x_4$ for $p = 1, 2, 3, 4$, and the process may be continued so that finally we have

$$b_p = \lambda Y_p + \frac{\partial b}{\partial x_p} \tag{3}$$

for all values of p.

Again, from (1),

$$\frac{\partial a_p}{\partial x_q} - \frac{\partial a_q}{\partial x_p} = \frac{\partial b}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial b}{\partial x_q}\frac{\partial Y}{\partial x_p}$$

$$= \frac{\partial}{\partial x_p}\left(b\frac{\partial Y}{\partial x_q}\right) - \frac{\partial}{\partial x_q}\left(b\frac{\partial Y}{\partial x_p}\right),$$

or

$$\frac{\partial}{\partial x_q}\left(a_p + b\frac{\partial Y}{\partial x_p}\right) = \frac{\partial}{\partial x_p}\left(a_q + \frac{\partial Y}{\partial x_q}\right),$$

and therefore

$$a_p = -b\frac{\partial Y}{\partial x_p} + \frac{\partial Z}{\partial x_p},$$

where Z is a new function. The complete solution of (1) is therefore given by

$$a_p = -b\frac{\partial Y}{\partial x_p} + \frac{\partial Z}{\partial x_p}, \quad b_p = \lambda\frac{\partial Y}{\partial x_p} + \frac{\partial b}{\partial x_p}, \tag{4}$$

where Z, b, λ, are three arbitrary functions.

Now consider the equation

$$(5) \qquad \frac{\partial}{\partial x_r}(H_{pq}) + K_{pq}\frac{\partial Y}{\partial x_r} = \frac{\partial}{\partial x_q}(H_{pr}) + K_{pr}\frac{\partial Y}{\partial x_q}.$$

Keep p fixed, and let $H_{pq} = a_q$, $K_{pq} = -b_q$. We now have equation (1), and hence

$$(6) \qquad H_{pq} = -B_p\frac{\partial Y}{\partial x_q} + \frac{\partial Z_p}{\partial x_q},$$

$$(7) \qquad K_{pq} = \lambda_p\frac{\partial Y}{\partial x_q} + \frac{\partial B_p}{\partial x_q},$$

where λ_p, B_p, Z_p, denote $3n$ as yet arbitrary functions.

Again, $H_{pq} = H_{qp}$, and therefore from (6)

$$\frac{\partial Z_p}{\partial x_q} - \frac{\partial Z_q}{\partial x_p} = B_p\frac{\partial Y}{\partial x_q} - B_q\frac{\partial Y}{\partial x_p}.$$

This equation is of the same type as (1), and hence

$$(8) \qquad Z_p = -B\frac{\partial Y}{\partial x_p} + \frac{\partial C}{\partial x_p}, \quad B_p = \nu\frac{\partial Y}{\partial x_p} + \frac{\partial B}{\partial x_p}.$$

Similarly from the condition $K_{pq} = K_{qp}$ we have the equations

$$(9) \qquad B_p = -\lambda\frac{\partial Y}{\partial x_p} + \frac{\partial \eta}{\partial x_p}, \quad -\lambda_p = -\mu\frac{\partial Y}{\partial x_p} + \frac{\partial \lambda}{\partial x_p}.$$

It follows without difficulty that $\nu = -\lambda$, $\eta = B$, and hence, substituting in (6) and (7) we have the final results

$$H_{pq} = \lambda\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial B}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial B}{\partial x_p}\frac{\partial Y}{\partial x_q} - B\frac{\partial^2 Y}{\partial x_p \partial x_q} + \frac{\partial^2 C}{\partial x_p \partial x_q},$$

$$K_{pq} = \mu\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial \lambda}{\partial x_p}\frac{\partial Y}{\partial x_q} - \frac{\partial \lambda}{\partial x_q}\frac{\partial Y}{\partial x_p} - \lambda\frac{\partial^2 Y}{\partial x_p \partial x_q} + \frac{\partial^2 B}{\partial x_p \partial x_q}.$$

The $n(n+1)$ quantities H, K, thus depend on the four arbitrary functions λ, μ, B, C.

The above relations may also be written

$$H_{pq} = \frac{\partial^2 A}{\partial x_p \partial x_q} + Y\frac{\partial^2 B}{\partial x_p \partial x_q} + \lambda\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q},$$

$$K_{pq} = \frac{\partial^2}{\partial x_p \partial x_q}(B - \lambda Y) + Y\frac{\partial^2 \lambda}{\partial x_p \partial x_q} + \mu\frac{\partial Y}{\partial x_p}\frac{\partial Y}{\partial x_q}.$$

BRYN MAWR,
May, 1909.

NECESSARY CONDITIONS THAT THREE OR MORE PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER SHALL HAVE COMMON SOLUTIONS.

BY PROFESSOR C. A. NOBLE.

(Read before the San Francisco Section of the American Mathematical Society, September 26, 1908.)

In a paper inspired by Hilbert's lectures in 1900, Yoshiye (*Mathematische Annalen*, volume 57) considers, among others, the following problem in the calculus of variations: To find the necessary conditions that the integral

$$\int_{u_0}^{u_1} [\lambda(z' - px' - qy') + \mu(p' - rx' - sy') + \nu(q' - sx' - ty')]du$$

shall vanish independently of the path of integration, whereby the two equations

$$F(x, y, z, p, q, r, s, t) = 0, \quad G(x, y, z, p, q, r, s, t) = 0$$

shall be satisfied. λ, μ, ν are arbitrary functions of u; p, q, r, s, t have the usual signification, i. e., $p = \partial z/\partial x$, $q = \partial z/\partial y$, etc.; the accents denote differentiation with respect to u.

The conditions which result upon consideration of this problem are that the two equations

$$\nu^2 F_r + \mu\nu F_s + \mu^2 F_t = 0, \quad \nu^2 G_r + \mu\nu G_s + \mu^2 G_t = 0$$

shall have a common solution in $\mu : \nu$, i. e., that the determinant

$$\begin{vmatrix} F_r & F_s & F_t & 0 \\ G_r & G_s & G_t & 0 \\ 0 & F_r & F_s & F_t \\ 0 & G_r & G_s & G_t \end{vmatrix}$$

shall vanish; and furthermore, that for the value of $\mu : \nu$ which satisfies these two equations the relation

$$\mu[F_t(G_y) - G_t(F_y)] + \nu[F_r(G_x) - G_r(F_x)] = 0$$

shall be satisfied. F_x, F_y, etc., in the foregoing, denote par-

tial derivatives with respect to x, y, etc.; the symbols (F_x), (F_y) are abbreviations for

$$F_x + F_z p + F_p r + F_q s \quad \text{and} \quad F_y + F_z q + F_p s + F_q t$$

respectively, and similarly for (G_x) and (G_y).

If one adds to the auxiliary conditions, in Yoshiye's problem, the third partial differential equation

$$H(x, y, z, p, q, r, s, t) = 0,$$

the first variation of the following integral must vanish:

$$\int_{u_0}^{u_1} [\lambda(z' - px' - qy') + \mu(p' - rx' - sy') + \nu(q' - sx' - ty') + \xi F + \eta G + \zeta H]\, du$$

where ξ, η, ζ are Lagrange multipliers. The Lagrange equations for the determination of the functions x, y, z, p, q, r, s, t, λ, μ, ν, ξ, η, ζ are as follows:

1) $(\lambda p)' + (\mu r)' + (\nu s)' + \xi F_x + \eta G_x + \zeta H_x = 0,$

2) $(\lambda q)' + (\mu s)' + (\nu t)' + \xi F_y + \eta G_y + \zeta H_y = 0,$

3) $\lambda' + \xi F_z + \eta G_z + \zeta H_z = 0,$

4) $\mu' - \xi F_p - \eta G_p - \zeta H_p + \lambda x' = 0,$

5) $\nu' - \xi F_q - \eta G_q - \zeta H_q + \lambda y' = 0,$

6) $\mu x' - \xi F_r - \eta G_r - \zeta H_r = 0,$

7) $\mu y' + \nu x' - \xi F_s - \eta G_s - \zeta H_s = 0,$

8) $\nu y' - \xi F_t - \eta G_t - \zeta H_t = 0,$

9) $z' - px' - qy' = 0,$ 10) $p' - rx' - sy' = 0,$

11) $q' - sx' - ty' = 0,$

12) $F = 0,$ 13) $G = 0,$ 14) $H = 0,$

Eliminating λ', μ', ν' from 1) and 2) by means of 3), 4), and 5), we obtain

15) $\mu r' + \nu s' + \xi(F_x) + \eta(G_x) + \zeta(H_x) = 0,$

16) $\mu s' + \nu t' + \xi(F_y) + \eta(G_y) + \zeta(H_y) = 0.$

Substituting in

$$\frac{dF}{du} = F_x x' + F_y y' + F_z z' + F_p p' + F_q q' + F_r r' + F_s s' + F_t t' = 0$$

for $x', y', z', p', q', r', t'$ their values taken from 6), 8), 9), 10), 11), 15), 16), we obtain

$$\frac{\xi F_r + \eta G_r + \zeta H_r}{\mu}(F_x) + \frac{\xi F_t + \eta G_t + \zeta H_t}{\nu}(F_y)$$
$$- F_r \cdot \frac{\nu s' + \xi(F_x) + \eta(G_x) + \zeta(H_x)}{\mu} + F_s s'$$
$$- F_t \cdot \frac{\mu s' + \xi(F_y) + \eta(G) + \zeta(H_y)}{\nu} = 0.$$

Two similar equations arise when we employ $dG/du = 0$ and $dH/du = 0$. Solving in these three equations for s', and using the further abbreviation

$$\mu[F_t(G_y) - G_t(F_y)] + \nu[F_r(G_x) - G_r(F_x)] \equiv (F, G),$$

whereby $(F, F) \equiv 0$ and $(F, G) = -(G, F)$, we obtain

$$\alpha) s'\{\nu^2 F_r - \mu\nu F_s + \mu^2 G_t\} = (F, F)\xi + (G, F)\eta + (H, F)\zeta,$$
$$\beta) s'\{\nu^2 G_r - \mu\nu G_s + \mu^2 G_t\} = (F, G)\xi + (G, G)\eta + (H, G)\zeta,$$
$$\gamma) s'\{\nu^2 H_r - \mu\nu H_s + \mu^2 H_t\} = (F, H)\xi + (G, H)\eta + (H, H)\zeta.$$

From the last three equations we can deduce the following: Since $F = 0$ and $G = 0$ must have solutions in common, the left hand members of $\alpha)$, $\beta)$ must have a common root in $\mu : \nu$; and for this common root (F, G) must vanish. It follows that for this same $\mu : \nu$ (H, F) and (H, G) also vanish, hence also that both members of $\gamma)$ vanish. In other words, the left hand members of $\alpha)$, $\beta)$, $\gamma)$ have a common root in $\mu : \nu$, the condition for which is

$$\begin{Vmatrix} F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 \\ 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 \\ 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t \\ G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 \\ 0 & G_r & G_s & G_t & 0 & H_r & H_s & H_t \end{Vmatrix} = 0.$$

For this common root in $\mu : \nu$ we have $(F, G) = 0, (G, H) = 0$ and $(H, F) = 0$, which yields

$$\frac{F_r(G_x) - G_r(F_x)}{F_t(G_y) - G_t(F_y)} = \frac{G_r(H_x) - H_r(G_x)}{G_t(H_y) - H_t(G_y)} = \frac{H_r(F_x) - F_r(H_x)}{H_t(F_y) - F_t(H_y)}.$$

If we inquire as to the possibility of common solutions to the four partial differential equations of second order

$$F = 0, \quad G = 0, \quad H = 0, \quad K = 0,$$

we obtain, by analogous procedure, the following four equations

$$\alpha') \quad s'\{\nu^2 F_r - \mu\nu F_s + \mu^2 F_t\} = (F, F)\xi + (G, F)\eta + (H, F)\zeta + (K, F)\theta,$$

$$\beta') \quad s'\{\nu^2 G_r - \mu\nu G_s + \mu^2 G_t\} = (F, G)\xi + (G, G)\eta + (H, G)\zeta + (K, G)\theta,$$

$$\gamma') \quad s'\{\nu^2 H_r - \mu\nu H_s + \mu^2 H_t\} = (F, H)\xi + (G, H)\eta + (H, H)\zeta + (K, H)\theta,$$

$$\delta') \quad s'\{\nu^2 K_r - \mu\nu K_s + \mu^2 K_t\} = (F, K)\xi + (G, K)\eta + (H, K)\zeta + (K, K)\theta.$$

Since $F = 0$, $G = 0$, and $H = 0$ have common solutions, the left hand members (and hence the right hand members) of α'), β'), γ') vanish for the same $\mu : \nu$; and for this $\mu : \nu$ we have

$$(F, G) = 0, \quad (G, H) = 0, \quad (H, F) = 0,$$

consequently also

$$(K, F) = 0, \quad (K, G) = 0, \quad (K, H) = 0;$$

and hence the left hand member of δ') vanishes for this same $\mu : \nu$. We obtain therefore, as a necessary condition that

$$F = 0, \quad G = 0, \quad H = 0, \quad K = 0$$

may have common solutions

$$\begin{vmatrix} F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t \\ G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 & K_r & K_s & K_t & 0 \\ 0 & G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 & K_r & K_s & K_t \end{vmatrix} = 0.$$

And since

$$(F, G) = (F, H) = (F, K) = (G, H) = (G, K) = (HK) = 0$$

for the same value of $\mu : \nu$, we have, in addition to the vanishing of the above matrix,

$$\frac{F_r(G_x) - G_r(F_x)}{F_t(G_y) - G_t(F_y)} = \frac{F_r(H_x) - H_r(F_x)}{F_t(H_y) - H_t(F_y)} = \frac{F_r(K_x) - K_r(F_x)}{F_t(K_y) - K_t(F_y)}$$

$$= \frac{G_r(H_x) - H_r(G_x)}{G_t(H_y) - H_t(G_y)} = \frac{G_r(K_x) - K_r(G_x)}{G_t(K_y) - K_t(G_y)} = \frac{H_r(K_x) - K_r(H_x)}{H_t(K_y) - K_t(H_y)}.$$

Obviously the plan is general, and one could write down the necessary conditions that a system of n partial differential equations of the type above considered should have solutions in common.

NOTE ON DETERMINANTS WHOSE TERMS ARE CERTAIN INTEGRALS.

BY PROFESSOR R. G. D. RICHARDSON AND MR. W. A. HURWITZ.

(Read before the American Mathematical Society, September 14, 1909.)

THE object of the present note is to prove two simple identities involving a determinant whose elements are certain integrals, and to mention some special cases. Determinants of the form considered present themselves in problems connected with linear differential and integral equations and the calculus of

variations.* The theorems proved include in particular the well-known Schwarz's and Bessel's identities and inequalities, which are of value in the theory of linear integral equations.† A determinant involving infinite series in the same way in which the determinants here considered involve integrals is used by Schmidt in his theory of points and vectors of a function space.‡ Expressions of the same character present themselves in the problem of finding, according to the criterion of the method of least squares, the nearest approximation to an arbitrary function in terms of given functions, not necessarily orthogonal.

1. *Let* $\phi_1, \phi_2, \cdots, \phi_n$; $\psi_1, \psi_2, \cdots, \psi_n$ *denote functions which are limited and integrable in a certain region* R *of an* m*-dimensional space, or if unlimited, are such that the product of any function of the first set by any function of the second set is integrable; let* D_i, $\overline{D}_j$ *denote the determinant obtained by replacing respectively the* i*th row of the determinant*

$$D = \begin{vmatrix} \int_R \phi_1\psi_1 dR & \int_R \phi_1\psi_2 dR & \cdots & \int_R \phi_1\psi_n dR \\ \int_R \phi_2\psi_1 dR & \int_R \phi_2\psi_2 dR & \cdots & \int_R \phi_2\psi_n dR \\ \cdots & \cdots & \cdots & \cdots \\ \int_R \phi_n\psi_1 dR & \int_R \phi_n\psi_2 dR & \cdots & \int_R \phi_n\psi_n dR \end{vmatrix}$$

by $\psi_1, \psi_2, \cdots, \psi_n$ *or the* j*th column by* $\phi_1, \phi_2, \cdots, \phi_n$, *and let* Δ_{ij} *be the co-factor of that term of* D *which stands in the* i*th row and the* j*th column. Then*

$$(1) \qquad \int_R D_i \overline{D}_j dR = D\Delta_{ij}.$$

* For further identities which arise when the functions involved are known to be solutions of certain differential equations, see an article by Professor Richardson entitled "Das Jacobische Kriterium der Variationsrechnung und die Oscillationseigenschaften linearer Differentialgleichungen 2. Ordnung," shortly to appear in the *Mathematische Annalen*.

† Cf. Erhard Schmidt, "Zur Theorie der linearen und nichtlinearen Integralgleichungen," *Mathematische Annalen*, vol. 63 (1907), p. 433.

‡ "Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten," *Rendiconti del Circolo Matematico di Palermo*, vol. 25 (1908), p. 62.

The proof follows directly from the following applications of well-known theorems on the expansion of determinants:

$$\sum_{q=1}^{n}\Delta_{rq}\int_R \phi_p\psi_q dR = \begin{cases} 0, & p \neq r, \\ D, & p = r, \end{cases}$$

$$\sum_{q=1}^{n}\Delta_{iq}\psi_q = D_i, \quad \sum_{p=1}^{n}\Delta_{pj}\phi_p = \bar{D}_j.$$

For we have

$$\int_R D_i\bar{D}_j dR = \sum_{p=1}^{n}\sum_{q=1}^{n}\Delta_{iq}\Delta_{pj}\int_R \phi_p\psi_q dR$$

$$= \sum_{p=1}^{n}\Delta_{pj}\Big[\sum\Delta_{iq}\int_R \phi_p\psi_q dR\Big] = \Delta_{ij}D,$$

which was to be proved.

2. *Let $R_1, R_2, \cdots, R_n$ denote n equal regions in different m-dimensional spaces and $\phi_i(R_j)$, $\psi_i(R_j)$ the functions ϕ_i, ψ_i defined in the region R_j. Then*

$$(2) \qquad \int_{R_1}\cdots\int_{R_n}\begin{vmatrix}\phi_1(R_1) & \cdots & \phi_1(R_n)\\ \cdot & \cdots & \cdot\\ \phi_n(R_1) & \cdots & \phi_n(R_n)\end{vmatrix}\cdot\begin{vmatrix}\psi_1(R_1) & \cdots & \psi_1(R_n)\\ \cdot & \cdots & \cdot\\ \psi_n(R_1) & \cdots & \psi_n(R_n)\end{vmatrix} \times dR_1\cdots dR_n = n!\,D.$$

In order to prove the theorem we note that the sign of any term $\phi_{i_1}(R_{k_1})\cdot\phi_{i_2}(R_{k_2})\cdots\phi_{i_n}(R_{k_n})$ in the first determinant of the integrand, where each of the sequences $i_1, i_2, \cdots, i_n$; $k_1, k_2, \cdots, k_n$ is a permutation of the sequence of integers $1, 2, \cdots, n$, is the power of -1 whose index is the number of transpositions required to build the substitution $\left(\begin{smallmatrix} i_1 i_2 \cdots i_n \\ k_1 k_2 \cdots k_n \end{smallmatrix}\right)$ — that is, to bring the term considered into the principal diagonal. Similarly the sign of the term $\psi_{j_1}(R_{k_1})\cdot\psi_{j_2}(R_{k_2})\cdots\psi_{j_n}(R_{k_n})$ in the second determinant is the power of -1 whose index is the number of transpositions required to build the substitution $\left(\begin{smallmatrix} j_1 j_2 \cdots j_n \\ k_1 k_2 \cdots k_n \end{smallmatrix}\right)$. The result of integrating the product of these two terms is

$$\int_{R_{k_1}}\phi_{i_1}(R_{k_1})\psi_{j_1}(R_{k_1})dR_{k_1}\cdots\int_{R_{k_n}}\phi_{i_n}(R_{k_n})\psi_{j_n}(R_{k_n})dR_{k_n}$$

$$= \int_R \phi_{i_1}\psi_{j_1}dR\cdots\int_R\phi_{i_n}\psi_{j_n}dR,$$

to which is attached a sign equal to the power of -1 whose index is the sum of the numbers of transpositions required to build the substitutions $\left(\begin{smallmatrix} i_1 i_2 \cdots i_n \\ k_1 k_2 \cdots k_n \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} j_1 j_2 \cdots j_n \\ k_1 k_2 \cdots k_n \end{smallmatrix}\right)$; this sum furthermore is equal to the number of transpositions required to build the substitution $\left(\begin{smallmatrix} i_1 i_2 \cdots i_n \\ j_1 j_2 \cdots j_n \end{smallmatrix}\right)$. The process of integration therefore leads to a term of D with the proper sign attached. By using every permutation of the sequence $(1, 2, \cdots, n)$ for each of $(i_1, i_2, \cdots, i_n)$, $(j_1, j_2, \cdots, j_n)$, $(k_1, k_2, \cdots, k_n)$ we obtain the result of integrating every term in the product of the two determinants on the left of (2). By using every permutation of this sequence for each of $(i_1, i_2, \cdots, i_n)$, $(j_1, j_2, \cdots, j_n)$, we obtain every term of D once and only once for each choice of $(k_1, k_2, \cdots, k_n)$. We have thus the value of D repeated as often as the number of the permutations of the sequence $(1, 2, \cdots, n)$, — that is, $n!$ times — which proves the theorem.

3. Both theorems reduce to interesting identities under various specializations. For example, if we take* $\psi_1 = k\phi_1$, $\psi_2 = k\phi_2$, $\cdots$, $\psi_n = k\phi_n$, we have from (1)

$$(3)\quad D\Delta_{11} = \int_R k \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \int k\phi_2\phi_1 dR & \int k\phi_2^2 dR & \cdots & \int k\phi_2\phi_n dR \\ \cdots & \cdots & \cdots & \cdots \\ \int k\phi_n\phi_1 dR & \int k\phi_n\phi_2 dR & \cdots & \int k\phi_n^2 dR \end{vmatrix}^2 dR,$$

where

$$D = \begin{vmatrix} \int k\phi_1^2 dR & \int k\phi_1\phi_2 dR & \cdots & \int k\phi_1\phi_n dR \\ \int k\phi_2\phi_1 dR & \int k\phi_2^2 dR & \cdots & \int k\phi_2\phi_n dR \\ \cdots & \cdots & \cdots & \cdots \\ \int k\phi_n\phi_1 dR & \int k\phi_n\phi_2 dR & \cdots & \int k\phi_n^2 dR \end{vmatrix},$$

$$\Delta_{11} = \begin{vmatrix} \int k\phi_2^2 dR & \cdots & \int k\phi_2\phi_n dR \\ \cdots & \cdots & \cdots \\ \int k\phi_n\phi_2 dR & \cdots & \int k\phi_n^2 dR \end{vmatrix};$$

* In this case the restriction imposed above reduces to the condition that $k\phi_i\phi_j$ is integrable $(i = 1, 2, \cdots, n\,;\ j = 1, 2, \cdots, n)$.

and from (2)

$$(4) \qquad D=\frac{1}{n!}\int_{R_1}\cdots\int_{R_n} k(R_1)\cdots k(R_n) \times \begin{vmatrix} \phi_1(R_1) & \cdots & \phi_1(R_n) \\ \cdots & \cdots & \cdots \\ \phi_n(R_1) & \cdots & \phi_n(R_n) \end{vmatrix}^2 dR_1\cdots dR_n.$$

It follows that, if k is nowhere negative in R, D is not negative, and that if k is everywhere positive in R, D cannot vanish unless the functions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly dependent. This result is obviously a generalization of the theorem known as Schwarz's inequality, in which the case $m=1$, $n=2$ is considered.

4. Let us consider further two sets of functions $\phi_1, \phi_2, \cdots, \phi_n$; $\psi_1, \psi_2, \cdots, \psi_n$ satisfying the relations*

$$\int_R \phi_i\psi_j dR = 0 \quad (i=1, 2, \cdots, n;\ j=1, 2, \cdots, n;\ i\neq j).$$

If we write (1) for the sets of $n+1$ functions $\phi_0=f, \phi_1, \phi_2, \cdots, \phi_n$; $\psi_0=g, \psi_1, \psi_2, \cdots, \psi_n$, we have for $i=j=0$ the result

$$(5) \qquad \int_R\left[Af-\sum_{p=1}^{n} A_p\phi_p\int_R f\psi_p dR\right] \times\left[Ag-\sum_{p=1}^{n} A_p\psi_p\int_R g\phi_p dR\right]dR = A\left[A\int_R fg dR-\sum_{p=1}^{n} A_p\int_R f\psi_p dR\int_R g\phi_p dR\right],$$

where

$$a_p=\int_R \phi_p\psi_p dR, \quad A=a_1\cdot a_2\cdots a_n,$$

$$A_p=a_1\cdot a_2\cdots a_{p-1}\cdot a_{p+1}\cdots a_n.$$

The further assumption that $\phi_i=\psi_i$ and that $a_i=1$ $(i=1, 2, \cdots, n)$, i. e., that we are dealing with a normalized set

* E. g., solutions of the differential equations $L(\phi_i)+\lambda_i\phi_i=0$, $M(\psi_j)+\lambda_j(\psi_j)=0$ (where L and M are adjoint differential expressions of the second order in any number of independent variables) in a closed region R, which vanish on the boundary of R.

of orthogonal functions, leads, if applied to the case $f = g$ to Bessel's identity

$$(6)\quad \int_R \Big[f - \sum_{p=1}^{n} \phi_p \int_R f\phi_p dR \Big]^2 dR = \int_R f^2 dR - \sum_{p=1}^{n} \Big[\int_R f\phi_p dR \Big]^2,$$

from which Bessel's inequality immediately follows.

5. Theorems analogous to those of the present note involving finite sums or infinite series in place of integrals may be proved in a similar manner.

GÖTTINGEN,
June, 1909.

ON THE TACTICAL PROBLEM OF STEINER.

BY PROFESSOR W. H. BUSSEY.

(Read before the American Mathematical Society, February 24, 1906.)

THE study of tactical configurations known as triple systems had its origin in two problems proposed independently by J. Steiner * and T. P. Kirkman. † The Steiner problem, which is the more general and includes the other, is as follows:

For what values of n is it possible to arrange n elements in sets of three, called triads, so that every set of two elements is contained in one and only one triad? If n is a number for which there is such an arrangement in triads, are there other arrangements that cannot be obtained from it by a mere permutation of the elements? When such an arrangement in triads has been made, is it possible to arrange the n elements in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three that is not a triad is contained in one and only one tetrad? When such an arrangement in tetrads has been made, is it possible to arrange the n elements in sets of five, called pentads, so that no triad or tetrad is contained in a pentad, and so that every set of four that is not a tetrad and does not contain a triad is contained in one and only one pentad? In general, when an arrangement in k-*ads* has been made, is it possible to arrange the n elements in sets of $k+1$, called $(k+1)$-*ads* so that no l-*ad* $(l \leqq k)$ is contained in a $(k+1)$-*ad*, and so that every set of k elements that is not a

* *Journal für die reine und angewandte Mathematik*, vol. 45, p. 181.

† *The Lady's and Gentleman's Diary* for 1850. For other references to the literature of Kirkman's fifteen school girls problem see Ball's Mathematical Recreations and Essays, 4th edition, page 121.

k-ad and does not contain an *l-ad* ($l < k$) is contained in one and only one $(k+1)$-*ad*?

The part of the problem that relates to triads has been completely solved. * The other parts have been little studied.

If an arrangement of n elements in triads, tetrads, pentads, etc., is possible, the number of *k-ads* for $k = 3, 4, 5, \cdots$ is given by the formula

$$N_k = \frac{1}{k!} n(n-1)(n-3)\cdots(n-[2^{k-2}-1]).$$

This formula was suggested by Steiner. It may be proved without much difficulty by complete induction.

This paper has to do with the case in which n is a number of the form $2^j - 1$. Its object is to show that it is possible to arrange such a number of elements in *k-ads* for $k = 3, 4, 5, \cdots, j+1$. The formula gives $N_k = 0$ when $k > j+1$.

Consider the $2^{k+1} - 1$ elements $(x_1, x_2, x_3, \cdots, x_{k+1})$, each x being 0 or 1 and the element $(0, 0, 0, \cdots, 0)$ being excluded. For convenience the language of geometry is used and each of the elements is called a point. The $2^{k+1} - 1$ points are said to constitute a finite geometry of k dimensions, or, more briefly, a k-space.† Consider also the linear homogeneous congruence, modulo 2,

$$(1) \qquad a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{k+1}x_{k+1} \equiv 0,$$

in which each coefficient is 0 or 1 and at least one of them is not zero. The points of the k-space that satisfy such a congruence are said to constitute a $(k-1)$-space; the points that satisfy two linearly independent congruences of the type (1) are said to constitute a $(k-2)$-space; and, in general, the points that satisfy $(k-l)$ linearly independent congruences of type (1) are said to constitute an l-space. The number of solutions of a set of congruences of type (1) may be counted without much difficulty and the number of points in an l-space, $l < k$, found to be $2^{l+1} - 1$. In particular, the number of points in a plane (2-space) is seven, and the number in a line (1-space) is three. A single point constitutes a 0-space. The points common to two l-spaces, if there are any, constitute an r-space, where $0 \leqq r \leqq l-1$. A set of $l+1$ points which are

* *Encyclopédie des Sciences mathématiques*, vol. 1, p. 80.

†See Veblen and Bussey, "Finite projective geometries," *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 241–259. In particular, see § 2.

not all contained in the same $(l-1)$-space is contained in one and only one l-space. The $l+1$ points of such a set, if taken l at a time, determine a number of $(l-1)$-spaces whose points constitute a set that may conveniently be called a *simplex* * of order l. The $l+1$ points are called vertices. A convenient symbol for a simplex of order l is $S(l+1)$. Any $i+1$ of the vertices of a $S(l+1)$ are the vertices of a simplex $S(i+1)$ whose points are all contained in the $S(l+1)$.

THEOREM. *The number of points in a simplex of order l is one less than the number of points in the l-space determined by its $l+1$ vertices.*

By actual count, the theorem is true for $l \leqq 3$. The rest of the proof consists in showing that it can be proved for a simplex $S(m+1)$ if it be assumed true for every simplex $S(l+1)$ for which $l < m$. This is done by arranging the points of the simplex $S(m+1)$ in the m following sets. The sets are not mutually exclusive.

1. The $m+1$ vertices of the simplex $S(m+1)$.

2. The points of the ${}_{m+1}C_2$ lines determined by the vertices taken two at a time.

3. The points of the ${}_{m+1}C_3$ planes determined by the vertices taken three at a time.

4. The points of the ${}_{m+1}C_4$ 3-spaces determined by the vertices taken four at a time.

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$i+1$. The points of the ${}_{m+1}C_{i+1}$ i-spaces determined by the vertices taken $i+1$ at a time.

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n. The points of the ${}_{m+1}C_m$ $(m-1)$-spaces determined by the vertices taken m at a time.

[Note: The symbol ${}_{m+1}C_j$ means the number of combinations of $m+1$ things taken j at a time.]

The set numbered $i+1$, i being any one of the numbers $1, 2, 3, \cdots, m$, consists of the points contained in ${}_{m+1}C_{i+1}$ i-spaces each of which is determined by $i+1$ of the vertices of the simplex $S(m+1)$ or, in other words, by the $i+1$ vertices of a simplex $S(i+1)$ which is contained in the simplex $S(m+1)$. By hypothesis, each of these i-spaces contains one and only one

*The word is used in geometry of n-dimensions to denote the configuration analogous to the triangle in the plane or the tetrahedron in 3-space.

point not contained in the simplex $S(i+1)$ that determines it. But that one point is a point of the simplex $S(m+1)$ by the very definition of simplex. Therefore, if one begins to count with the first set and counts through the sets in order, the number of points in the set numbered $i+1$ that have not been counted in any previous set is ${}_{m+1}C_{i+1}$. It follows that the number of points in the simplex $S(m+1)$ is

$$\sum_{j=1}^{m} {}_{m+1}C_j = 2^{m+1} - 2,$$

which is one less than the number of points in the m-space determined by the $m+1$ vertices of the simplex.

From this theorem it follows that the $l+1$ vertices of a simplex of order l determine uniquely another point, namely, the one point of the l-space determined by the simplex that is not also a point of the simplex. It is convenient to call this point the point complementary to the simplex. The triads, tetrads, pentads, etc. of the Steiner problem are found as follows: Every simplex $S(2)$ determines a triad consisting of its two vertices and the complementary point; every simplex $S(3)$ determines a tetrad consisting of its three vertices and the complementary point; and, in general, every simplex $S(l-1)$, $l \leqq k+2$, determines an *l-ad* consisting of the $l-1$ vertices and the complementary point. There are no *l-ads* for $l > k+2$.

When $n = 2^6 - 1 = 63$, it is possible to arrange the n elements in triads, tetrads, pentads, hexads, and heptads. There is no arrangement of the 63 elements in l-ads for $l > 7$. This special case was involved in Steiner's investigation of the configuration of the 28 double tangents of a quartic curve * and led him to propose for solution the "Combinatorische Aufgabe" which I have called "The tactical problem of Steiner."

ON THE SO-CALLED GYROSTATIC EFFECT.

BY PROFESSOR ALEXANDER S. CHESSIN.

(Read before the American Mathematical Society, April 24, 1909.)

In computing the resisting couple of gyrostats or the so-called "gyrostatic effect" it is customary to assume that it is equal to $C\lambda\omega \sin\theta$, where C, λ, ω and θ denote respectively the moment of inertia of the gyrostat about its geometrical axis, the angular

* *Journal für die reine und angewandte Mathematik*, vol. 49, pp. 265–272.

velocity of spin, the angular velocity of precession and the angle of the geometrical axis with the axis of precession. It is proposed here to give the exact value of this couple.

3. The gyrostatic effect is due to the action of what I have called the *convective* and the *turning* forces in the motion of a body relatively to a moving system (XYZ).* Let ω be the angular velocity of this system; p, q, r its components along X, Y, Z. To simplify the results we will assume that the center of gravity of the body is at the origin of these axes. Then the principal moments M and M' of the convective and the turning forces are given by their components

$$M_x = \sum_i (y_i F_{iz} - z_i F_{iy}), \quad M'_x = \sum_i (y_i F'_{iz} - z_i F'_{iy}),$$

$$M_y = \sum_i (z_i F_{ix} - x_i F_{iz}), \quad M'_y = \sum_i (z_i F'_{ix} - x_i F'_{iz}),$$

$$M_z = \sum_i (x_i F_{iy} - y_i F_{ix}), \quad M'_z = \sum_i (x_i F'_{iy} - y_i F'_{ix}),$$

where

$$\frac{1}{m_i} F_{ix} = z_i \frac{dq}{dt} - y_i \frac{dr}{dt} + p(px_i + qy_i + rz_i) - \omega^2 x_i,$$

$$\frac{1}{m_i} F_{iy} = x_i \frac{dr}{dt} - z_i \frac{dp}{dt} + q(px_i + qy_i + rz_i) - \omega^2 y_i,$$

$$\frac{1}{m_i} F_{iz} = y_i \frac{dp}{dt} - x_i \frac{dq}{dt} + r(px_i + qy_i + rz_i) - \omega^2 z_i,$$

$$\frac{1}{m_i} F'_{ix} = -2(q\dot{z}_i - r\dot{y}_i), \quad \frac{1}{m_i} F'_{iy} = -2(r\dot{x}_i - p\dot{z}_i),$$

$$\frac{1}{m_i} F'_{iz} = -2(p\dot{y}_i - q\dot{x}_i).$$

Let now (ΞHZ) be a system of axes coinciding with the principal axes of inertia of the body.

By a series of transformations which it does not seem worth

* "On relative motion," *Transactions American Mathematical Society*, vol. 1.

while to reproduce here we obtain the following expressions for M and M':

$$M_\xi = A\dot{\omega}_\xi - (B - C)\omega_\eta\omega_\zeta, \quad M_\eta = B\dot{\omega}_\eta - (C - A)\omega_\zeta\omega_\xi,$$
$$M_\zeta = C\dot{\omega}_\zeta - (A - B)\omega_\xi\omega_\eta,$$
$$M'_\xi = (A + B - C)Q\omega_\zeta - (A - B + C)R\omega_\eta,$$
$$M'_\eta = (B + C - A)R\omega_\xi - (B - C + A)P\omega_\zeta,$$
$$M'_\zeta = (C + A - B)P\omega_\eta - (C - A + B)Q\omega_\xi,$$

where A, B, C and P, Q, R are the principal moments of inertia of the body and the components of its angular velocity Ω in the relative motion.

When two of the moments of inertia, as in the case of gyrostats, are equal, *i. e.*, $A = B$, the expressions given above will be simplified by the following selection of axes: axis P, trace of plane ΞH on plane XY; axis Q at right angle to P in the plane ΞH (to the right of P relatively to R); and R coincident with Z. Introducing Euler's angles θ, ϕ, ψ we shall now have

$$M_1 = A\dot{\omega}_1 - (A - C)\omega_2\omega_3, \quad M'_1 = (2A - C)\omega_3\phi' \sin\theta - CR\omega_2,$$
$$M_2 = A\dot{\omega}_2 - (C - A)\omega_3\omega_1, \quad M'_2 = -(2A - C)\omega_3\theta' + CR\omega_1,$$
$$M_3 = C\dot{\omega}_3, \qquad M'_3 = C(\omega_2\theta' - \omega_1\phi' \sin\theta),$$

using the symbols 1, 2, 3 in lieu of P, Q, R to indicate the corresponding components.

Were we to select the axes (XYZ) so that the Z axis coincide with the axis (ω), we would have $\omega_1 = 0$, $\omega_2 = \omega \sin\theta$, $\omega_3 = \omega \cos\theta$, and therefore

$$M'_1 = (2A - C)\omega\phi' \sin\theta\cos\theta - C\omega R \sin\theta,$$
$$M'_2 = -(2A - C)\omega\theta' \cos\theta, \quad M'_3 = C\omega\theta' \sin\theta.$$

These results show that even assuming that the system (XYZ) revolves with a constant angular velocity and that this velocity (ω) is very small (so that we may neglect terms of the order of ω^2), the commonly accepted value of the gyrostatic couple is incorrect. The last formulas reduce to $|M'| = C\omega\lambda \sin\theta$ if we assume that $\theta' = \phi' = 0$, *i. e.*, that the gyrostat axis is invariably fixed in the system (XYZ).

A CONTINUOUS GROUP RELATED TO VON SEIDEL'S OPTICAL THEORY.

BY DR. ARTHUR C. LUNN.

(Read before the Chicago Section of the American Mathematical Society, April 17, 1908.)

THE determination of the various aberrations of an axially symmetric optical instrument according to the method systematized by Petzval and von Seidel rests on the computation of certain power series whose coefficients are functions of what may be called the paraxial magnitudes of the system. These are the quantities which describe what would be the course of the rays of light through the ideal instrument, to which the actual instrument is an approximation, and which was defined by Abbe* as producing an exact collineative transformation of the object into the image. This ideal transformation coincides with the classic first approximation of geometric optics, a general analytic representation of which was given by Gauss,† together with a method of computation using continued fractions. A set of formulas more convenient in practice was introduced by von Seidel,‡ using as defining coordinates of a ray in a meridian plane the optical height, or distance of the ray from any point of the axis, measured along the normal to the latter, and the angle, or in the ideal collineation the tangent of the angle, between the ray and the axis.

If these coordinates be called x, θ respectively, von Seidel's paraxial equations are equivalent to

$$\Delta\mu\theta = xk\Delta\mu, \quad \Delta x = 0, \quad D\theta = 0, \quad Dx = -t\theta,$$

where μ is the refractive index of a medium, k the curvature of a refracting spherical surface, t the distance of transmission through a homogeneous medium; Δ, D denote the changes corresponding respectively to refraction at a surface where μ changes and to propagation through a medium of constant μ separating two consecutive surfaces. It will be shown here that these equations may be viewed as transformations generat-

* Czapski-Eppenstein, Theorie der optischen Instrumente nach Abbe, chap. ii.

† Dioptrische Untersuchungen, Ges. Werke, V, p. 243.

‡ *Astr. Nach.*, Nos. 1027-1029 (April, 1855).

ing a certain three-parameter group, whose properties thus admit of optical interpretation. In particular the determination of the complete system of invariants seems to mark the exact scope of possible application of Abbe's concept of "optical invariant" in the formulation of optical computations.

Let $\mu\theta = u$, then the transformations and their generating differentiators are

$$(1) \qquad u' = u + \alpha x, \quad x' = x, \quad U_1 = x\frac{\partial}{\partial u}, \quad (\alpha = k\Delta\mu)$$

$$(2) \qquad x' = x + \beta u, \quad u' = u, \quad U_2 = u\frac{\partial}{\partial x}, \quad \left(\beta = -\frac{t}{\mu}\right)$$

each of which defines a one-parameter group in the variables x, u, the respective parameters α, β being arbitrary because the equations apply to systems having arbitrary curvatures and axial spacing of the refracting surfaces.

The first group relates to successive refractions at a set of surfaces in contact, and corresponds to the ordinary theory of thin lenses close together. For example, the additive property of the focal powers of such lenses is the interpretation of the additive combination of the parameters α in the compounding of the transformations, while the invariance of x means simply that the primitive and refracted rays intersect at the refracting surface.

The second group relates to the optical effect of media stratified in parallel planes. The invariance of u means that such a system has no focal power according to the definition of Gauss, and the additive combination of parameters corresponds to the foreshortening effect of each stratum according to its thickness and refractive index.

The dual character of the two transformations under simultaneous interchange of x, u and α, β was recognized by von Seidel in his use of odd and even subscripts respectively for refracting surfaces and intervening media. It suggests its own phrasing for various statements. For instance, a principal focus is a point where $x = 0$ on a ray transformed from a primitive ray having $u = 0$, or parallel to the axis.

The successive commutators formed from U_1 and U_2 are

$$(3) \qquad (U_1U_2) = U_3 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad (U_1U_3) = -2U_1, \quad (U_2U_3) = 2U_2.$$

There is thus generated a three-parameter group whose equations may be obtained* by integration of

$$\frac{dx'}{dt} = \lambda_2 u' + \lambda_3 x', \quad \frac{du'}{dt} = \lambda_1 x' - \lambda_3 u',$$

giving

$$x' = A_1\lambda_2 e^{\omega t} + A_2\lambda_2 e^{-\omega t},$$
$$u' = A_1(\omega - \lambda_3)e^{\omega t} - A_2(\omega + \lambda_3)e^{-\omega t},$$
$$\omega^2 = \lambda_3^2 + \lambda_1\lambda_2,$$

where the constants A are to be determined in terms of the initial coordinates x, u. A convenient final form, containing the three parameters a, b, c is

$$(4) \qquad x' = xC + (cx + bu)S, \qquad u' = uC + (ax - cu)S,$$

where $C = \cosh\tau$, $S = \sinh\tau/\tau$, $\tau^2 = c^2 + ab$, and $a = \lambda_1 t$, $b = \lambda_2 t$, $c = \lambda_3 t$. The groups of U_1 and U_2 individually are obtained by putting respectively $b = c = \tau = 0$, $a = \alpha$, and $a = c = \tau = 0$, $b = \beta$.

The combination of two transformations such as (4) with parameters (a, b, c) and (a', b', c') is equivalent to a single such transformation with parameters (a'', b'', c'') determined by

$$\begin{aligned} C'' + c''S'' &= (C' + c'S')(C + cS) + ab'S'S, \\ b''S'' &= bS(C' + c'S') + b'S'(C - cS), \\ C'' - c''S'' &= (C' - c'S')(C - cS) + a'bS'S, \\ a''S'' &= aS(C' - c'S') + a'S'(C + cS), \end{aligned} \qquad (5)$$

which may be replaced by

$$\begin{aligned} C'' &= CC' + SS'\{cc' + \tfrac{1}{2}(ab' + a'b)\}, \\ c''S'' &= cSC' + c'S'C + \tfrac{1}{2}SS'(ab' - a'b), \\ b''S'' &= bSC' + b'S'C + SS'(bc' - b'c), \\ a''S'' &= aSC' + a'S'C + SS'(ca' - c'a). \end{aligned} \qquad (6)$$

Here the first equation determines the set of possible values of the auxiliary τ'', then a'', b'', c'' are given by the last three

* Lie-Engel, Theorie der Transformationsgruppen, I, p. 70.

equations; and a direct computation shows that if $\tau^2 = c^2 + ab$ and $\tau'^2 = c'^2 + a'b'$ then also $\tau''^2 = c''^2 + a''b''$, verifying the group property. The solution fails only if $S'' = 0$, which occurs when τ'' has the form $2n\pi i$ where n is an integer not zero. With τ in the neighborhood of such a value it is more convenient to use a/τ, b/τ, c/τ as parameters. It should be noticed that some complex values of the parameters give real transformations, and that if (a, b, c) and (a', b', c') are real numbers, still (a'', b'', c'') are not necessarily such. Moreover, (a, b, c) may be real but τ pure imaginary, if $c^2 + ab$ is negative.

The group of U_3 alone, obtained by putting $a = b = 0$, $c = \tau$, has the form

$$x' = xe^{\tau}, \quad u = ue^{-\tau}, \tag{7}$$

where the parameters combine additively and xu is invariant.

The general transformation (4) can be built up in the form $T_2T_1T'_2$, where each T_i is a transformation of the one-parameter group of U_i. For, if the factors have parameters β, α, β' respectively, the identification rests upon the solution of the equations

$$1 + \beta\alpha = C + cS, \quad \alpha = aS, \quad 1 + \beta'\alpha = C - cS,$$
$$\beta + \beta' + \beta\beta'\alpha = bS.$$

The first three give

$$\alpha = aS, \quad \beta = (C - 1 + cS)/aS, \quad \beta' = (C - 1 - cS)/aS,$$

and these values satisfy the fourth equation also, because of the relation $\tau^2 = c^2 + ab$. An alternative factorization is the dual form $T_1T_2T'_1$, with parameters α, β, α', obtained by interchange of α and β and also of a and b.

The first form interprets the general transformation in terms of a single refracting surface as giving the relation of the coordinates of a refracted ray at distance t' behind the surface to those of the primitive ray at distance t in front of the surface. A special case of interest is where x' is independent of u, or $\beta + \beta' + \beta\beta'\alpha = 0$. This is equivalent to $\mu/t + \mu'/t' = (\mu' - \mu)/r$, the ordinary equation of conjugate distances with respect to the surface. Such transformations occur for instance when the successive positions of a ray traversing any system are located with respect to the planes of the aperture diaphragm and its successive images.

The second form interprets the general transformation in terms of two separated refracting surfaces, and gives the relation of the coordinates of the primitive and refracted rays at points in the first and third media in contact with the surfaces; giving thus an analogy to the theorem of Gauss that any system is in the first approximation equivalent to a single thick lens. The special case, dual to the one mentioned in the preceding paragraph, is where u' is independent of x, or $\alpha + \alpha' + \alpha\alpha'\beta = 0$. This is equivalent to the condition that the second principal focus of the first surface coincides with the first of the second surface, so that the lens is "afocal" or telescopic.

The three-parameter group in two variables thus far considered has no invariant other than a constant. But in the estimation of certain optical errors, such as astigmatism and distortion, which depend on the location of the apertures limiting the pencils of rays, it is necessary to consider not only the systems of rays proceeding from the object and transformed into those converging upon the images formed by the successive refractions, but also a fictitious system of rays proceeding from the "entrance pupil" or initial aperture diaphragm, real or virtual, and refracted into those converging upon its successive images in the same optical system. If the rays of this second system be specified by coordinates v, y, then these will be variables transformed cogrediently with u, x, by equations similar to (4) with the same parameters, belonging thus to a three-parameter group in the four variables generated by

$$U_1 = x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}, \quad U_2 = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y},$$

$$U_3 = (U_1 U_2) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$

so that a fundamental system will consist of a single proper invariant.

To find its form, let $\rho = ux + vy$, $\sigma = uy - vx$. Then the U's expressed in terms of the variables (x, y, ρ, σ) are

$$U_1 = (x^2 + y^2)\frac{\partial}{\partial \rho}, \quad U_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$U_2 = \frac{1}{(x^2 + y^2)}\left\{(\rho x + \sigma y)\frac{\partial}{\partial x} + (\rho y - \sigma x)\frac{\partial}{\partial y} + (\rho^2 + \sigma^2)\frac{\partial}{\partial \rho}\right\},$$

whose form shows directly that an invariant must be independent of x, y, ρ, and therefore a function of σ only.

If u, x belong to a ray passing through the margin of the image and the axial point of the diaphragm, and v, y similarly to a ray passing through the center of the image and the edge of the diaphragm, a geometric construction shows that the expression $uy - vx$ is equivalent to the product of refractive index, lateral radius of image, and tangent of angular semi-aperture of central pencil. The invariant σ is therefore identical with the expression which occurs in the well-known equation pointed out in a special case by Lagrange, but first found as general by Helmholtz.* The result here shows that this is essentially the only relation expressible in terms of invariants which gives a general property of the paraxial transformation.

CHICAGO. ILL.,
April, 1909.

SHORTER NOTICES.

Analytische Geometrie der Ebene. By C. RUNGE. Leipzig, B. G. Teubner, 1908. 198 pp.

SHALL we regard elementary analytic geometry, the analytic geometry we teach in a first course to freshmen or sophomores, as a body of doctrine with which it is useful for the student to become acquainted, or shall we rather regard it as an instrument with whose use he is to be made so familiar that he shall always be ready to employ it even in a quite new problem? This is a question which every teacher of analytic geometry and every writer of a text-book on the subject is called upon to face. Upon its answer the nature of the text-book written or selected for use will depend. Few persons, it is true, would go to such an extreme as to adopt without qualification either of the views above referred to. Those who regard it as their main object to inculcate a beautiful and important doctrine would deem it essential that the student gain the power of making the application of the general theorems learned to concrete cases, and the teacher who regards elementary analytic geometry primarily as a method whose use is to be taught would not neglect the opportunity of explaining

* Handbuch der physiologischen Optik, 1. Auflage, p. 50.

and illustrating the use of this method, so far as possible, by means of problems which are in themselves of lasting interest and importance. With this reservation, however, the above classification serves to separate teachers and writers of text-books fairly well into two opposing camps.

In the former camp it is customary to take as one of the largest and most essential parts of the body of doctrine to be inculcated the theory of the conic sections; and the subjects of conjugate diameters, poles and polars, the general equation of the second degree, etc., form essential chapters with whose contents the student is supposed to become familiar. On the other hand such a teacher will usually be content, in the main, with numerical problems, which serve to test the pupil's understanding of the formulas and principles developed.

The teacher who adopts the second standpoint will also usually contrive, as has been suggested above, to familiarize his students with many facts concerning conic sections; but he will do this incidentally, his main concern being all the time to train his student to do things for himself. He will, therefore, regard the numerical problem as the lowest and least useful of all problems (except, indeed, for the pupil incapable of rising higher), while the problem which requires the student to handle his instrument in a slightly new way will be regarded as the highest type, to be used sparingly on account of its difficulty, but as invaluable because it gives to the good student the very best training possible in becoming a real master in the use of his method.

There is still a third current which has been running very strongly of late both in this country and in England, which, so far as it is concerned with the subject with which we are here dealing, has as its motto: "Analytic geometry is a necessary evil. Let us have as little of it as possible." The devotees of this cult, who commonly regard all mathematics as merely a tool, naturally take the same point of view with regard to analytic geometry, but, being engineers or in the employ of engineers, they believe that they have no interest in anything beyond the very simplest geometrical relations, and that consequently this particular tool needs only very slight development at their hands. In the rudimentary condition in which they are willing to leave it, numerical problems are about all that one can venture upon, and this is all they ask.

Professor Runge's book has an individuality and interest of

its own and refuses to fit in perfectly to the pigeon-holes of any classification. This is just what one familiar with his other writings would expect; and yet some of his statements have at first a very familiar ring. We quote the opening words of the preface:

"This book has grown out of lectures which for many years I was in the habit of delivering at the technological school at Hannover, and in writing it I have had constantly in mind the needs of the engineer. It seems to me that the analytic geometry taught in technological schools should, first of all, be a tool for following up geometrical relations by means of numerical computations."

The programme here laid out is carried through, so far as the simplest properties of the straight line and circle are concerned, in Part II, which extends from page 20 to page 63. This part is perhaps the most distinctive and interesting of the whole book. The problems taken up are of the very simplest types: A line is determined by two points, to compute the ordinates of those points on this line which have given abscissas. Or again: To compute the coordinates of the point of intersection of two lines each of which is determined by a point through which it passes and the angle it makes with the axis of x. Each of these problems (they are the first two, and are typical) is treated at length by several different methods, four or five pages being devoted to each problem. This seems surprising until on closer examination we notice that questions of analytic geometry proper are here quite pushed into the background by pure questions of convenient forms of numerical computation. What method will be best if logarithms are to be used? If we use the slide-rule? If we use a computing machine? How can the numerical work be best arranged? These are all good questions, and we Americans may well wish that systematic courses on numerical computation were more frequent in this country. It seems, however, to the reviewer that such questions should hardly be considered at the very beginning of a course on analytic geometry, as they are then in danger of overshadowing the fundamental principles of the subject.

Perhaps, however, in spite of its title and general appearance, it is hardly fair to treat this book as one from which the beginner will get his first acquaintance with analytic geometry. It should rather be regarded as supplementing in a most interesting way the more traditional treatises from which students

will continue to derive most of their working knowledge. Moreover the insistence on the numerical side of the subject, to which we have referred, is only one feature of the book. Long sections of a decidedly theoretical character follow in which affine and perspective transformations of the plane play an important part. Indeed the ellipse, hyperbola, and parabola are introduced as the images of the circle under these transformations, and the whole theory of these curves is made to depend on this point of view. Homogeneous coordinates are also treated at length in the last chapter.

On the whole the book departs less essentially than one would at first suppose from the traditional German text-book in which the subject is presented rather from the point of view of *kennen* than *können*. It is still a body of doctrine which is presented, though the choice of subjects is somewhat unusual, including as it does, besides the subjects already mentioned, a section on the computation of stresses in frameworks of light rods, and an introduction to some of the most elementary aspects of the use of vectors. Even the questions of numerical computation may fairly be regarded as part of this scheme rather than as an attempt to put the student on his own mettle. To a teacher collecting material for a course on numerical computation the section to which reference has just been made will be found most useful. MAXIME BÔCHER.

Gruppen- und Substitutionentheorie. By Dr. EUGEN NETTO. Sammlung Schubert LV. G. J. Göschen, Leipzig, 1908. viii + 175 pp.

IN conformity with the general plan of the Sammlung Schubert Professor Netto aims to give in this book an introduction to the theory of groups of finite order. He has succeeded admirably in his purpose. Those readers who are not already familiar with the details of the theory will find Chapter II particularly valuable in fixing for them the fundamental notions of the subject, if they take the pains to work through the details. Indeed we do not know if there is another place where this particular phase of the subject is treated so happily. But this is by no means the only good chapter. They are all excellent and the book as a whole is a fine example of clear and attractive exposition.

The author introduces some new notation which is of value in the interest of brevity. On page 35 the greatest common sub-

group of the groups $G, H, K, \cdots$ is denoted by the symbol $\}G, H, K, \cdots\{$. The largest subgroup of a group G within which a given subgroup J of G is invariant is called the "Zwischengruppe" of J in G (page 54). New notations in the interest of precision of statement are found on page 58.

The book is not free from misprints, some of which we note. In lines 8 and 7 from the bottom of page 45 H and G should be replaced by H and Γ respectively. The inequality sign in line 7 from the bottom of page 88 should be reversed. The c_λ on page 102, line 3, should be replaced by σ_λ. In line 5 from the bottom of page 103 J_1 should be in the place of H_1. The latter part of the first formula on page 169 should be

$$(p^\kappa - 1)\, p^{\kappa-1}\,(p^{\kappa-1} - 1)\, p^{\kappa-2} r_{\kappa-2}.$$

Netto's definition of a group (page 1) is exactly Dickson's definition of a semigroup.* If only a finite number of elements are under consideration, as is the case practically throughout the book, the two notions coincide. But some of the illustrations of groups given on page 2 are, according to Dickson, and also according to de Séguier,† not groups at all, but semigroups.

The book contains a few slight inaccuracies that should be noticed. The statement at the beginning of § 116 applies obviously only to groups of composite order. Moreover, the word transitive in this statement is redundant. The summing up of § 54, page 71, should not apply to the factors of composition $r/r_1, r_1/r_2, \cdots, r_{\nu-1}/r_\nu$. The statement near the bottom of page 102 that there is a $(p^\beta, 1)$ isomorphism between G and Γ is not true in general. The $(s, 1)$ isomorphism that is mentioned near the bottom of page 139 in reality exists between G and D, and not between G and S, as the author states it. As is well known, there are five primitive groups of degree 5. In the enumeration of these on page 156, the cyclic group is omitted.

The definition given of the *class* of a substitution group (page 128) implies that this term is applied only to k-fold transitive groups, where $k>2$. This is not the definition given by Jordan.‡

The statement on page 123, in regard to solvable groups of

* *Trans. Amer. Math. Soc.*, vol. 6 (1905), p. 205.

† Éléments de la théorie des groupes abstraits, p. 8.

‡ *Liouville's Journal*, ser. 2, vol. 16 (1871), p. 408. A somewhat different definition is given by Jordan in the *Comptes Rendus* of the Paris Academy, vol. 73 (1871), p. 853.

order $p_1^{a_1}p_2^{a_2}$ could, in view of Burnside's results,* have been made much more general.

This enumeration of a few points in which we think the book might be improved should not be understood as detracting from our statement at the beginning of this review commending the book. In publishing so excellent a treatment of the subject Professor Netto has performed a service of value to the mathematical public. W. B. FITE.

Einführung in die höhere Mathematik. Von EMANUEL CZUBER. Teubner, 1909. 382 pp.

THIS book is an amplification of the lectures on differential and integral calculus given by Professor Czuber at the technical school in Vienna. The treatment has been extended so as to form a good introduction to higher mathematics, adapted for students other than technical students. The subjects are developed with much care and rigor.

There are essentially three divisions, viz., Functions of real variables, Algebra, and Analytical geometry. The opening chapter is on real and imaginary numbers. The number concept is developed for use in the functions of a real variable. The second chapter is a short, concise, and elegant presentation of infinite series and products. Besides simple demonstrations the subject is made easier for the student by the many well-chosen examples to illustrate the points in question.

Chapters III and IV begin the theory of functions of real variables. The general idea of a function, limit of a function, and continuity are the principal topics discussed. Here again we find many well-chosen examples. The function

$$f(x) = \lim_{n=\infty} \frac{nx + 2}{nx^2 + 1}$$

is given to show the difference between the value of a function given by direct substitution and that obtained by the limiting process. The substitution of $x = 0$ gives $f(0) = 2$, while if we first proceed to the limit and then put $x = 0$ we have $f(0) = \pm\infty$.

The example

$$f(x) = \frac{x}{1 + e^{1/x}}$$

is a good illustration of a function which is continuous for $x = 0$, but whose derivative is discontinuous at this point.

* *Proceedings Lond. Math. Society*, series 2, vol. 1 (1904), p. 388.

$f(x) = x[x]$, ($[x]$ indicating the largest integer contained in x), is an example of a function which is discontinuous at points for which x is an integer, but whose derivative is continuous.

$f(x) = x[1/x]$ is an example of a function which is discontinuous at points where x is an integer and whose derivative is also discontinuous at these points.

The chapter on the applications of the differential calculus contains nothing more than the applications to indeterminate forms and maxima and minima. It seems rather unfortunate that Taylor's and Maclaurin's series should be omitted entirely.

The part of the book dealing with algebra contains a chapter of forty pages on determinants, and a chapter on algebraic equations which treats of resultants, discriminants, and the solution and discussion of numerical equations.

The last 140 pages are devoted to analytical geometry. It is rather remarkable to note that geometric loci are discussed on the fourth page, the equations of the conics, strophoid, cissoid, Cassinian ovals, and four cusped hypocycloid being derived as examples in loci. The particular equations of the line and conic are then taken up and discussed in detail. The treatment of analytical geometry is satisfactory indeed.

The book as a whole is well adapted to the purpose for which it was written, but as is usually the case with the European text it does not contain a sufficient number of exercises and problems which are left for the student. Throughout there are many footnotes, mostly of a historical nature, which are sufficient to arouse an interest in the history of the subject.

C. L. E. MOORE.

Récréations Mathématiques et Problèmes des Temps Anciens et Modernes. Par W. W. ROUSE BALL. Deuxième édition française traduite d'après la quatrième édition anglaise et enrichie de nombreuses additions par J. FITZ-PATRICK. Paris, A. Hermann, 1907. 8vo. 3 parts. 5 francs each.

THE subject of mathematical recreations has always occupied a prominent position in the history of science. Zeno, Alcuin, Bachet, Fermat, Lucas, — these are only a few of the hundreds of names that might be mentioned of those who have contributed to this interesting field. Many of these men have been mathematicians of no small repute, for in reality the border line between recreative and serious mathematics is purely imaginary. To the mathematician all mathematics is a recreation; it is to

him what color is to Sorolla or form to Michelangelo or rhythm to a reader of Poe, and the teacher is a poor one who does not appreciate this fact. It is because mathematics is itself a subject full of interesting situations, of wonder, and of rhythm that more students enjoy it than some of our pedagogical agitators think, and it is because of this that much of the effort to humanize mathematics to-day is really, though well meant, an effort to make it less human.

It is partly for those mistaken teachers who feel that mathematics has not the same interest per se that music or art or literature has, that Mr. Ball prepared this interesting collection upwards of seventeen years ago. But it was also for the mathematician himself, who abuses his nerve system in his love for the more serious side of the science, that the book was written, even as Punch is published not merely for the casual reader but also for the statesman who needs to see his labors in a different light after a night in parliament.

The English work has now passed through four editions and the French translation through two, which testifies anew to the pleasant style and to the wisdom of selection that characterizes Mr. Ball's various publications.

The French edition is considerably more extended than the English original in some respects, filling three volumes. The chief departure from the original is in Chapter I, Some arithmetical questions. This occupied less than forty pages in the English edition, but it makes up the first volume, of over three hundred pages, in the translation. The added material relates largely to the history of numbers and to interesting problems of early and medieval times. The mysticism of numbers, so exhaustively treated from the religious side by Bungus three centuries ago, speculations on the platonic number, curious properties of decimal numbers, the application of algebra to number games, and the elementary theory of numbers in general are some of the features of the French edition that make it well worth placing upon the shelves of any mathematical or general library.

DAVID EUGENE SMITH.

Lehrbuch der Kristalloptik. Von F. POCKELS. Teubners Sammlung XIX. Leipzig, B. G. Teubner, 1906. x + 520 pp.

ALTHOUGH the past five years have seen the publication of a large number of books on optics, the subject is so broad and

may be treated from so many points of view that there has been relatively little duplication in the different works and there is still room for other treatments. Where one book may lay the stress on the analytical theory and be replete with formulas, another may take the physical side and with relatively few formulas establish the chief points of relation between optics and electromagnetism, and yet another may deal with optics from what might be called a dynamical point of view. Pockels's book is none of these, but as its name indicates is concerned with crystalline optics; in fact it will offer much more of interest to the crystallographer than to the mathematician. Yet so thoroughly has the author covered his field and so encyclopedic is his treatment, that any student of optics, whether primarily interested in crystals or not, will find frequent occasion to consult the work, if only to ascertain what may be the known experimental facts with regard to the subject.

After a short introduction, the author divides his work into four major parts which treat respectively transparent crystals without rotary properties, crystals with rotary properties, absorbing crystals, and effects of external influences upon optical properties. Of these divisions the first is naturally the longest and indeed contains more than half of all the matter in the book. From the very introduction the presentation is true to the main object of the author, namely, to describe the phenomena of crystalline optics rather than to construct a well-knit theory. The start is made, not with theories, but with a few statements concerning the propagation of light, the wave surface, rays and plane waves, and Huyghens's principle from which many of the essentials of crystalline optics may be obtained without the need of intricate formulas or detailed physical considerations. Thereupon follow definitions of the light vector, of natural and polarized light, and the presentation of trigonometric and exponential representations of the light vector.

The method of the introduction is pursued during the first two chapters of the first main part of the work. The propagation of light in uniaxial crystals is described with the aid of the wave surface and related surfaces derived from Huyghens's principle. Here, as throughout the book, tables of the optical constants of the type of crystals considered are freely introduced. Even the phenomenon of dispersion with an appropriate table of data is mentioned at this early stage. The second chapter is almost an exact parallel of the first, but treats

biaxial crystals. Thus without any complicated analytical or physical investigations and in the short space of about seventy pages the author has succeeded in giving the reader an excellent account of the general question of the propagation of light in crystals and of the geometric method of discussing the propagation.

The third chapter on the theoretical physical foundations for the theory of light is a model of excellent and concise presentation of a difficult and often confused subject. The trouble is that there are so many different theories of light which give results differing very slightly, usually so slightly that there is no crucial experiment sufficiently accurate to distinguish between them. The elastic theory with its various developments is first mentioned. But it is the electromagnetic theory which naturally comes in for the major part of the discussion. In the propagation of electromagnetic waves there are four vectors, the magnetic induction B, the magnetic force H, the electric induction or displacement D, and the electric force E which may possibly be taken as the light vector. Of these the first two B and H are so nearly parallel in bodies which propagate light that it would be hopeless to distinguish between them and useless to consider as different the theories built upon them. Although many theorists use these vectors, Pockels discards them in favor of the electric vectors, as would be expected in view of results obtained from experiments with stationary waves. As between D and E, the author chooses D as his light vector. He then has a few words to say about the electron theories with especial reference to their bearing on dispersion. Although Pockels thus introduces a little physical optics into his work, the mere fact that Zeeman's name does not occur here or elsewhere is sufficient to show how strictly he adheres to his aim of treating the optics of crystals and how carefully he avoids being led off into the general realm of physical optics.

The fourth chapter is on reflection and refraction. To show the detail with which the subject is presented it will be sufficient to mention the titles of the articles on total reflection. They are: general conditions, methods of observation, the limiting cone of rays in the case of reflection from a uniaxial crystal, polarization of the limiting rays, special cases of total reflection from biaxial crystals, singular phenomena connected with conical refraction, determination of optical constants for

biaxial crystals by means of observations on total reflection. It would scarcely be possible to find a more detailed or simpler presentation of all this material. Many works on analytical or physical optics hardly mention these subjects at all except in the most general way. Several chapters further on in the work, the author takes up the physical basis for these results and develops the formulas from the electromagnetic conditions at the interface of two media.

There is little need of prolonging this review with the recitation of the course of the various chapters. Enough has been said to show the method upon which the author has constructed his book and the detail with which he has written. There is one point in which an improvement might be suggested. The plates which exhibit the elaborate and intricate phenomena of interference are all in black and white. This is a great pity; the beautifully modulated color schemes are the chief attraction of the figures and the author's detailed tables of the colors that are found in some special cases by no means take the place of the actual colors on the plates. If the coloring of the plates had to be done by the eye and hand, there might be good excuses for omitting it; but natural color photography is now so well developed that very good photographs of these effects can be obtained and reproduced.

The student of optics, who frequently finds it very hard to lay his hand upon a large and accurate presentation of the phenomena of crystalline optics, will refer constantly to this work and will feel under deep obligations to its author for the pains taken in preparing it.

E. B. WILSON.

NOTES.

THE July number (volume 10, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Projective differential geometry of curved surfaces (fifth memoir)," by E. J. WILCZYNSKI; "On the osculating quartic of a plane curve," by W. W. DENTON; "Note on a system of axioms for geometry," by A. R. SCHWEITZER; "Irreducible homogeneous linear groups in an arbitrary domain," by W. B. FITE; "On the integration of the homogeneous linear difference equation of second order," by W. B. FORD; "On Cantor's theorem concerning the coefficients of a convergent

trigonometric series, with generalizations," by W. F. OSGOOD; "Equivalence of pairs of bilinear or quadratic forms under rational transformation," by L. E. DICKSON; "On a complete system of invariants of two triangles," by D. D. LEIB.

THE July number (volume 31, number 3) of the *American Journal of Mathematics* contains the following papers: "The birational transformations of algebraic curves of genus 4," by A. L. VAN BENSCHOTEN; "On some loci associated with plane curves," by C. H. SISAM; "Plane sections of a Weddle surface," by F. MORLEY and J. R. CONNER; "The differential equation satisfying abelian theta functions of genus 3," by J. E. WRIGHT; "Differential equations admitting a given group," by J. E. WRIGHT; "On the angles of the regular polytopes of four dimensional space," by P. H. SCHOUTE.

THE concluding (July) number of volume 10 of the *Annals of Mathematics* contains: "Thermodynamic analogies for a simple dynamical system," by E. B. WILSON; "Discussion of a method for finding numerical square roots," by C. L. BOUTON; "On the direct product in the theory of finite groups," by J. H. MACLAGAN-WEDDERBURN; "Existence of the generalized Green's function," by W. D. A. WESTFALL; "The gambler's ruin," by J. L. COOLIDGE.

THE series of lectures announced in connection with the celebration, September 6–18, of the twentieth anniversary of the founding of Clark University included the following in mathematics: Professor E. H. MOORE, "The role of postulational methods in mathematics;" Professor E. B. VAN VLECK, "The homogeneous linear difference equation of order n with polynomial coefficients, considered from the functional standpoint;" Professor JAMES PIERPONT, "Modern theories of integration" Discussions of the following topics were also announced: "The unification and continuity of mathematics in school and college;" "The effectiveness of mathematical training," opened by Professor J. W. A. YOUNG; "The use and abuse of textbooks in mathematical classes," opened by Dr. J. S. FRENCH.

THE next meeting of the British association for the advancement of science will be held at Sheffield, England, August 31 to September 10, 1910, under the presidency of T. G. BONNEY.

THE section of mathematics and physics of the royal society of Naples announces the following prize problem:

"A systematic exposition of the known concepts regarding configurations of planes and of spaces, putting them into closer relation with the theory of substitutions, and including, if possible, some new contributions."

The value of the prize is 1000 lire. Competing memoirs must be written in Italian, Latin, or French, and be in the hands of the secretary of the academy before June 30, 1910.

THE prize of the Accademia dei Lincei of 10,000 lire, awarded for excellence in mathematical contributions, has been divided between Professor F. ENRIQUES, of the University of Bologna, and Professer T. LEVI CIVITA, of the University of Padua.

AN extensive laboratory for the construction of mathematical models and apparatus is being fitted up for the use of students of the Ecole Normale of the University of Paris, somewhat after the plan of that at Göttingen. It will be in charge of Professor E. BOREL.

ADVANCED courses in mathematics are announced for the winter semester 1909–1910 at the various German universities as follows:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Analytic geometry, four hours; Theory of analytic functions, II, four hours; Geometry of conics, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Theory of curves and surfaces, four hours; Applications of elliptic functions, four hours; Seminar, two hours. — By Professor G. HETTNER: Definite integrals, two hours. — By Professor J. KNOBLAUCH: Differential calculus, with exercises, five hours; Theory of elliptic functions, four hours. — By Professor R. LEHMANN-FILHÈS: Integral calculus, four hours; Determinants, four hours. — By DR. I. SCHUR: Theory of algebraic equations, four hours; Theory of linear differential equations, four hours.

UNIVERSITY OF LEIPZIG — By Professor C. NEUMANN: Theory of potential and spherical harmonics, four hours; Seminar, two hours. — By Professor O. HÖLDER: Elliptic functions, five hours; Theory of finite groups, one hour; Seminar, two hours. — By Professor K. ROHN: Analytic geometry of space, four hours; Descriptive geometry with exercises, four hours;

Seminar, two hours. — By Professor G. HERGLOTZ: Mechanics, four hours; Mechanics of continua, two hours; Seminar, two hours. — By Professor P. v. OETTINGEN: Elements of projective dioptics, one hour. — By Professor F. HAUSDORFF: Differential geometry with exercises, four hours; Algebraic numbers, two hours. — By Professor H. LIEBMANN: Differential and integral calculus with exercises, five hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Differential calculus, five hours; Analytic mechanics, four hours; Seminar on line and spherical geometry, two hours. — By Professor A. VOSS: Analytic geometry of space, four hours; Theory of algebraic curves, four hours; Seminar on theory of surfaces, two hours. — By Professor A. PRINGSHEIM: Introduction to analytic functions, five hours. — By Professor A. SOMMERFELD: Vector analysis, three hours; Thermo-dynamics, three hours; Seminar, two hours. — By Professor H. BRUNN: Modern development of analysis situs, two hours. — By Professor K. DOEHLEMANN; Descriptive geometry, with exercises, eight hours; Synthetic geometry, with exercises, five hours; Graphical representation, two hours. — By Dr. G. HARTOGS: Theory of abelian functions, four hours. — By Dr. O. PERRON: Advanced calculus, with exercises, five hours; Theory of continued fractions, two hours.

UNIVERSITY OF STRASSBURG. — By Professor H. WEBER: Differential and integral calculus, four hours; Calculus of variations, two hours; Seminar, two hours. — By Professor F. SCHUR: Analytic geometry, four hours; Selected chapters of the theory of surfaces, two hours; Seminar, two hours. — By Professor J. WELLSTEIN: Partial differential equations, five hours; Seminar, one hour. — By Professor M. SIMON: Methods in elementary mathematics, four hours. — By Professor S. EPSTEIN: Determinants, four hours.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1909–1910:

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELÀ: Calculus of variations, integral equations, and Laplace's series, three hours. — By Professor L. DONATI: Thermodynamics, kinetic theory of gases, magneto- and electro-optics, three hours. — By Professor L. PINCHERLE: Elliptic functions, integrals of algebraic differentials, and abelian functions, three hours.

UNIVERSITY OF CATANIA. — By Professor M. DE FRANCHIS: Geometry on algebraic surfaces, three hours. — By Professor G. LAURICELLA: Integral equations, development in series of characteristic functions, with application to vibrating strings and membranes, four and a half hours. — By Professor G. PENNACCHIETTI: Applications of elliptic functions to mechanical problems, four and a half hours. — By Professor C. SEVERINI: Selected topics in differential geometry, three hours.

INSTITUTE OF FLORENCE. — By Professor T. BOGGIO: Applications of integral equations to mathematical physics, three hours.

UNIVERSITY OF GENOA. — By Professor E. E. LEVI: Differential and integral equations, four hours. — By Professor G. LORIA: Theory of geometric transformations, three hours. — By Professor O. TEDONE: Problems of elastic equilibrium, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO: History of mathematics from Newton to Lagrange, three hours. — By Professor A. CAPELLI: Arithmetic theory of algebraic numbers, three hours. — By Professor R. MARCOLONGO: Hydrodynamics, three hours. — By Professor D. MONTESANO: Theory of geometric correspondences, four and a half hours. — By Professor E. PASCAL: Selected chapters in analysis, three hours. — By Professor L. PINTO: Electro-optics and Hertz's waves, four and a half hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS: Theory of functions and integral equations, four and a half hours. — By Professor U. CISOTTI: Mathematical theory of elasticity with technical application, three hours. — By Professor A. FAVARO: The life and work of Archimedes, three hours. — By Professor P. GAZZANIGA: Theory of numbers, three hours. — By Professor T. LEVI-CIVITA: Equations of dynamics and principles of celestial mechanics, four and a half hours. — By Professor G. RICCI: Absolute differential calculus, equilibrium and motion of solid elastic bodies, four hours. — By Professor F. SEVERI: Theory of continuous groups, three hours. — By Professor G. VERONESE: Synthetic geometry of hyperspace, four hours.

UNIVERSITY OF PALERMO. — By Professor G. BAGNERA: Automorphic functions, three hours. — By Professor M. GEB-

BIA: Propagation of heat, and thermodynamics, four and a half hours. — By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and a half hours. — By Professor A. VENTURI: Figures of planets, particularly of the Earth, with regard to elasticity, three hours.

UNIVERSITY OF PAVIA. — By Professor E. ALMANSI: Theory of potential, electrostatics and magnetism, three hours. — By Professor L. BERZOLARI: Geometry of hyperspace, three hours. — By Professor R. BONOLA: Imaginary in geometry, projective generation of certain curves and surfaces, linear systems of conics and quadrics, three hours. — By Professor F. GERBALDI: Functions of a complex variable and abelian integrals, three hours. — By Professor G. VIVANTI: Theory of algebraic numbers, three hours.

UNIVERSITY OF PISA. — By Professor E. BERTINI: Abelian integrals with application to the geometry on an algebraic curve, three hours. — By Professor E. BIANCHI: Calculus of variations and integral equations, four and a half hours. — By Professor U. DINI: Linear differential equations with application to the development of a given function in series, four and a half hours. — By Professor E. A. MAGGI: Advanced theoretic mechanics, Maxwell's theory of electro-magnetic fields, and electrons, four and a half hours. — By Professor P. PISSETTI: Figures and rotations of celestial bodies, and spherical astronomy, three hours.

UNIVERSITY OF ROME. — By Professor G. CASTELNUOVO: Abelian functions and geometric applications, three hours. — By Professor V. CERRUTI: Partial differential equations of the first order, three hours. — By Professor L. ORLANDO: Dynamics of balloons and of aëroplanes, three hours. — By Professor L. SILBERSTEIN: Complements of dynamics, electro-magnetic fields, and optics, three hours. — By Professor V. VOLTERRA: Integral and integro-differential equations with applications, four and a half hours; hydrodynamics and the theory of the tides, three hours.

UNIVERSITY OF TURIN. — By Professor G. PEANO: Mathematical logic, three hours. — By Professor G. SANNIA: Geometric applications of the calculus and intrinsic geometry, three hours. — By Professor C. SEGRE: Cubic surfaces and plane quartics, three hours. — By Professor C. SOMIGLIANA: Optics and electric oscillations, three hours.

THE University of Rochester has received, under the provisions of the will of the late Rear Admiral W. HARKNESS, professor of mathematics, U. S. Navy, almost his entire collection of astronomical instruments and a considerable part of his library.

PROFESSOR M. CANTOR, of the University of Heidelberg, has been elected associate member of the academy of sciences of Heidelberg.

PROFESSOR A. CAPELLI, of the University of Naples, Professor G. DARBOUX, of the University of Paris, and Professor Sir A. G. GREENHILL, formerly of the Ordnance College, Woolwich, have been elected foreign members of the royal institute of Venice.

PROFESSOR E. ALMANSI, of the University of Pavia, and Professor A. GARBASSO, of the University of Genoa, have been elected corresponding members of the Accademia dei Lincei.

PROFESSOR G. DARBOUX has been appointed the official delegate of the French government at the approaching Hudson-Fulton celebration.

THE title of Hofrat has been conferred upon Professor O. HÖLDER, of the University of Leipzig.

PROFESSOR L. MAURER, of the University of Tübingen, has been promoted to a full professorship of mathematics.

DR. R. E. v. MISES, of the technical school at Brünn, has been appointed associate professor of mathematics at the University of Strassburg.

PROFESSOR F. SCHUH, of Delft, has been appointed professor of analysis at the University of Groningen.

DR. C. E. WASTEELS has been appointed associate professor of rational mechanics at the University of Geneva.

PROFESSOR E. v. WEBER, of the University of Würzburg, has been promoted to a full professorship of mathematics.

DR. — BYDZOVSKY has been appointed docent in mathematics at the Bohemian University of Prague.

DR. T. J. I'A BROMWICH, of Cambridge University, has been promoted to an associate professorship of mathematics.

PROFESSOR F. PRYM, of the University of Würzburg, will retire from active service on October 1.

PROFESSOR A. GRÜNWALD, of the German technical school at Prague, has retired from regular teaching.

A MARBLE monument by G. Monteverde of the late Professor LUIGI CREMONA was unveiled with appropriate ceremonies at the engineering school of Rome on June 10.

PROFESSOR C. RUNGE, of the University of Göttingen, has been appointed Kaiser Wilhelm exchange professor of mathematics at Columbia University for the next academic year. The subject of his lectures will be "Graphical methods in physics and technic."

THE honorary degree of doctor of science has been conferred upon Professor E. H. MOORE, of the University of Chicago, by Yale University.

WILLIAMS COLLEGE conferred the degree of doctor of laws upon Professor H. B. FINE, of Princeton University, last June.

PROFESSOR R. MORRIS, of Rutgers College, has been promoted to a full professorship of mathematics and graphics.

PROFESSOR FLOYD FIELD, of the Georgia School of Technology, has been appointed full professor of mathematics and head of the department for 1909–1910.

AT Princeton University DR. E. SWIFT has been promoted to an assistant professorship of mathematics.

PROFESSOR C. N. HASKINS, of the University of Illinois, has accepted an assistant professorship of mathematics at Dartmouth College.

THE following changes are announced at the University of Illinois: Dr. C. H. SISAM, who returns after a year's study at Turin, has been promoted to an assistant professorship of mathematics. Dr. JACOB KUNZ, of the University of Michigan, has been appointed assistant professor of mathematical physics. During the coming year he will give courses in dynamics and in the theory of electrons. Dr. THOMAS BUCK, of the University of Chicago, has been appointed instructor in mathematics.

PROFESSOR W. A. MANNING, of Stanford University, has exchanged work with Mr. E. W. PONZER, of the University of Illinois, for the current academic year.

DR. J. H. MACLAGAN-WEDDERBURN has been appointed preceptor in mathematics at Princeton University.

DR. C. C. GROVE has been appointed instructor in mathematics at Columbia University.

DR. L. KARPINSKI, of the University of Michigan, will spend next year at study at Columbia University.

PROFESSOR SIMON NEWCOMB died at Washington, D. C., July 11, 1909. He was born in Nova Scotia March 12, 1835, and came to the United States when eighteen years old. For thirty-six years he was professor of mathematics at the United States Naval Academy, with especial duties at the Naval observatory at Washington, and for ten years he was professor of mathematics and astromony at Johns Hopkins University, and co-editor of the *American Journal of Mathematics*. For his extensive contributions to astronomy, celestial mechanics and in particular to the theory of the motion of the moon he was the recipient of a large number of honorary degrees, medals, and prizes. He was a member of the National academy of science, foreign member of the Royal society of London, foreign associate of the Institute of France, knight of the order "Pour le mérite" für Kunst und Wissenschaft, and member of many other societies and academies. He was a member of the AMERICAN MATHEMATICAL SOCIETY from 1891, and its president during 1896–1898.

AT the meeting of the Paris academy of sciences on August 18, a eulogy on Professor SIMON NEWCOMB, late foreign associate of the academy, was pronounced by the permanent secretary, Professor G. DARBOUX.

PROFESSOR V. CERRUTI, of the University of Rome, died August 20, 1909, at the age of 59 years. He was director of the School of Engineers, Senator of the Kingdom of Italy, and member of the Accademia dei Lincei and of the Italian society of sciences.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOLZA (O.). Vorlesungen über Variationsrechnung. Umgearbeitete und stark vermehrte deutsche Ausgabe der "Lectures on the calculus of variations." 2te Lieferung. Leipzig, Teubner, 1909. 8vo. 6 + 301–541 pp. M. 6.00

BRAND (E.). Ueber Tetraëder deren Kanten eine Fläche zweiter Ordnung berühren. (Diss.) Strassburg, 1908. 8vo. 50 pp.

BÜCHER, neue, über Naturwissenschaften und Mathematik. Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet. Mitgeteilt Frühjahr, 1909. Leipzig, Hinrich, 1909. 8vo. 23 pp. M. 0.30

CASTELNUOVO (G.). Lezioni di geometria analitica. 2a edizione. Roma-Milano, Segati e C., 1909. 8vo. 8 + 688 pp. L. 15.00

COOLIDGE (J. L.). The elements of non-euclidean geometry. Oxford, Clarendon, 1909. 8vo. 291 pp. Cloth. $5.00

DÖLP (H.). Grundzüge und Aufgaben der Differential- und Integralrechnung nebst den Resultaten. Neu bearbeitet von E. Netto. 12te Auflage. Giessen, Töpelmann, 1909. 8vo. 6 + 216 pp. Cloth. M. 1.80

DROYSEN (P.). Aufgaben aus der analytischen Geometrie. (Progr.) Belgard, 1909. 4to. 24 pp.

DUMAS (S.). Sur le développement des fonctions elliptiques en fractions continues. (Diss.) Zurich, 1908. 8vo. 50 pp.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française. Vol. II: Fonctions de variables réelles. Fascicule 1. Paris et Leipzig, Teubner, 1909. 8vo. 112 pp. M. 3.60

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III: Geometrie. C 5. Spezielle ebene algebraische Kurven, von G. Kohn und G. Loria. Teil 1: Ebene Kurven dritter und vierter Ordnung, von G. Kohn. Leipzig, Teubner, 1909. 8vo. Pp. 457–570.

FINKE (P.). Ueber Scharen von ∞^5 Kurven im gewöhnlichen Raume. (Diss.) Berlin, 1909. 8vo. 36 pp.

FUCHS (L.). Gesammelte mathematische Werke. Herausgegeben von R. Fuchs und L. Schlesinger. Band III. Abhandlungen (1888–1902) und Reden. Berlin, Mayer und Müller, 1909. 4to. 11 + 460 pp. M. 28.00

JAECKEL (W.). Geometrische Untersuchungen über Kurven auf Flächen dritter Ordnung mit vier Knotenpunkten. (Progr.) Ohlau, 1909. 4to. 22 pp.

KEMPF (A.). Tetraëder deren Kanten eine Fläche zweiter Ordnung berühren. (Diss.) Strassburg, 1908. 8vo. 46 pp.

KLEIN (F.). Elementarmathematik vom höheren Standpunkte aus. Teil II: Geometrie. Vorlesung, gehalten im Sommersemester 1908. Ausgearbeitet von E. Hellinger. Leipzig, Teubner, 1909, 8vo. 8 + 515 pp. M. 7.50

KOPPE (M.). Die Iteration des Sinus und anderer Funktionen. (Progr.) Berlin, Weidmann, 1909. 8vo. 28 pp. M. 1.00

KOWALEWSKI (G.). Einführung in die Determinantentheorie einschliesslich der unendlichen und der Fredholmschen Determinanten. Leipzig, Veit, 1909. 8vo. 5 + 550 pp. Cloth. M. 16.00

LEUTENEGGER (J.). Lehrbuch der Differential-Rechnung. Zum Gebrauche an höheren Lehranstalten sowie zum Selbststudium bearbeitet. Bern, Francke, 1909. 8vo. 160 pp. Cloth. M. 3.20

LIER (O.). Ueber Flächenscharen die durch Berührungstransformation in Kurvenscharen überführbar sind. (Diss.) Berlin, 1909. 8vo. 69 pp.

MEYER (W. F.). Allgemeine Formen- und Invariantentheorie. I Band, Binäre Formen. (Sammlung Schubert, XXXIII.) Leipzig, Göschen, 1909. 8vo. 8 + 376 pp. Cloth. M. 9.60

NETTO (E.). See DÖLP (H.).

SCHANZ (J.). Der Aufbau des komplexen Zahlengebiets auf der natürlichen Zahlenreihe. (Progr.) Berlin, Weidmann, 1909. 4to. 31 pp. M. 1.00

SCHWERING (K.). Lehrbuch der kleinsten Quadrate. Freiburg, Herder, 1909. 8vo. 8 + 105 pp. Cloth. M. 2.80

TRAMM (W.). Geometrische Diskussion des Hermite'schen Polynoms. (Diss.) Zürich, 1908. 4to. 32 pp.

II. ELEMENTARY MATHEMATICS.

BALTIN (R.) und MAIWALD (W.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie mit zahlreichen Anwendungen aus der Planimetrie und Physik für Seminare und Präparandenanstalten. Nach der Müller-Kutnewskyschen Aufgabensammlung bearbeitet. Teil II. 3te verbesserte Auflage. Leipzig, Teubner, 1909. 8vo. 8 + pp. 111–352. Cloth. M. 2.40

BASSI (A.). Esercizî e problemi di algebra complementare ad uso del secundo biennio degli istituti tecnici. Vol. I, parte 2. Bra, Rocca, 1909. 8vo. Pp. 145–318. L. 1.80

BOOLE (M.E.). Philosophy and fun of algebra. London, Daniel, 1909. 12mo. 90 pp. 2 s.

BRUNO (G. M.). Elementos de geometria, para la enseñanza secundaria y escuelas preparatorias. Paris, Bouret, 1909. 16mo. 400 pp.

COMBEROUSSE (C. de). Cours de mathématiques à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale supérieure, à l'Ecole centrale des arts et manufactures. 3e édition. Vol. 4: Algèbre supérieure. Paris, Gauthier-Villars, 1909. 8vo. 24 + 832 pp. Fr. 15.00

GRANVILLE (W. A.). Plane and spherical trigonometry and four-place tables of logarithms. New York, Ginn, 1909. 8vo. 11 + 264 + 38 pp. Cloth. $1.25

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HEINRICH (M.). Logarithmen-Tafeln, vierstellig und dreistellig, in neuer Anordnung. Berlin, Grote, 1909. 4to. 10 + 30 pp. Cloth. M. 1.00

KALENS (M.). Die Quadratur des Kreises. Ihre vollständige Auflösung. Neu bearbeitet. Saarbrücken, 1909. 8vo. 12 pp. M. 1.00

KOHL (A.) und ZSCHOCKELT (H.). Aufgaben für Rechnen und Geometrie zum Gebrauche in ländlichen Fortbildungsschulen. Leipzig, Hahn, 1909. 8vo. 3 + 76 pp. Boards. M. 0.50

KREUSCHNER (R.). Das Additionstheorem der Winkelfunktionen und die Aenderungen der Funktionswerte mit dem sich ändernden Argument am neuen Transporteur für Winkelfunktionen. (Progr.) Barmen, 1909. 4to. 11 pp.

LUCKE (F.). Einführung in die Goniometrie. (Progr.) Zerbst, 1909. 4to. 10 pp.

MAIWALD (W.). See BALTIN (R.).

MARTÍ ALPERA (F.). Las primeras lecciones de geometria. Burgos, Rodriguez, 1909. 8vo. 104 pp. P. 1.00

MATHEMATICAL questions and solutions. Edited by C. L. Marks. New Series, Vol. 15. London, Hodgson, 1909. 8vo. 6s. 6d.

PREMIÈRES NOTIONS d'algèbre avec de nombreux exercices à l'usage des écoles primaires; par une réunion de professeurs. Paris, Poussielgue, 1909. 16mo. 88 pp.

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WIENECKE (E.). Die Grundlehren der Planimetrie für Lehrerbildungsanstalten in genetischer Darstellung mit reichem Aufgabenmaterial. 2te Auflage. Berlin, Oehmigke, 1909. 8vo. 8 + 174 pp. Cloth. M. 3.50

WROBEL (E.). Uebungsbuch zur Arithmetik und Algebra, enthaltend die Formeln, Lehrsätze und Auflösungsmethoden in systematischer Anordnung und eine grosse Anzahl von Fragen und Aufgaben. Zum Gebrauche an Gymnasien, Realgymnasien und andern höheren Lehranstalten bearbeitet. Teil II. Rostock, Koch, 1909. 8vo. 12 + 484 pp. Cloth. M. 4.90

ZSCHOCKELT (H.). See KOHL (A.).

III. APPLIED MATHEMATICS.

COFFIN (J. G.). Vector analysis: an introduction to vector methods and their various applications to physics and mathematics. New York, Wiley, 1909. 12mo. 19 + 248 pp. $2.50

GRAF (E.). Technische Berechnungen für die Praxis des Maschinen- und Bautechnikers. Ein Handbuch über gelöste Beispiele aus der gesamten Mechanik, der Maschinen-, Holz- und Bautechnik, einschliesslich Eisenbeton- und Brückenbau. Leipzig, Barth, 1909. 8vo. 8 + 374 pp. Cloth. M. 6.80

HAVELOCK (T. H.). See JESSOP (C. M.).

JADANZA (N.). Trattato di geometria pratica. Torino, Bona, 1909. 8vo. 320 pp.

JESSOP (C. M.) and HAVELOCK (T. H.). Elementary mechanics. London, Bell, 1909. 12mo. 286 pp. 4s. 6d.

KÜSTER (F. W.). Logarithmische Rechentafeln für Chemiker, Pharmazeuten, Mediziner und Physiker. Im Einverständnis mit der Atomgewichtskommission der deutschen chemischen Gesellschaft für den Gebrauch im Unterrichtslaboratorium und in der Praxis berechnet. 9te vollständig neu berechnete Auflage. Leipzig, Veit, 1909. 8vo. 107 pp. Cloth. M. 2.40

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MAIDA (G.). Sopra un integrale primo delle equazioni del moto di un punto, il quale è cubico rispetto alle componenti della velocità: dissertazione di laurea in matematica. Palermo, Virzì, 1909. 8vo. 28 pp.

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